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ON THE PARITY OF $p(n,3)$ AND $p_\Psi(n,3)$

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Abstract. A partition of n is *relatively prime* if its parts form a relatively prime set. The number of partitions of n into exactly k parts is denoted by $p(n,k)$ and the number of relatively prime partitions into exactly k parts is denoted by $p_\Psi(n,k)$. In this paper we deal with the parities of $p(n,3)$ and $p_\Psi(n,3)$.

1. Introduction

Let k and n be positive integers such that $k \leq n$. Let $p(n)$ be the number of partitions of n and let $p(n,k)$ be the number of such partitions having exactly k parts. A standard reference on the theory of partitions is Andrews [1]. It is well known that the generating function for $p(n,k)$ is

$$\sum_{n\geq k} p(n,k)q^n = \frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)}. \tag{1.1}$$

Thus, for $k=2$ and $k=3$ as is well known we have

$$p(n,2) = \lfloor n/2 \rfloor \quad \text{and} \quad p(n,3) = \langle n^2/12 \rangle,$$

where $\lfloor x \rfloor$ denotes the floor of x and $\langle x \rangle$ denotes the integer closest to x. However, for $k \geq 4$ the formulas are not so neat, see for instance [2].

Let $p_\Psi(n)$ be the number of relatively prime partitions of n and let $p_\Psi(n,k)$ be the number of such partitions having exactly k parts. We note that the sequence $p_\Psi(n)$ starting as follows for $n \geq 1$,

$$1, 1, 2, 3, 6, 7, 14, 17, 27, 34, 55, 63, 100, 119, 167, 209, 296, 347, 489, 582,$$

is sequence **A000837** in Sloane's online encyclopedia of integer sequences [4]. To find $p_\Psi(n)$ and $p_\Psi(n,k)$, we can combine (1.1) with the following theorem.

Theorem 1.1.

$$p_\Psi(n) = \sum_{d|n} \mu(d) p(n/d), \tag{1.2}$$

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and

$$p_\Psi(n,k) = \sum_{d\mid n} \mu(d) p(n/d,k), \tag{1.3}$$

where $\mu(d)$ is the Möbius function.

Proof. For (1.2), we have

$$p(n) = \sum_{d\mid n} p_\Psi(n/d),$$

which by the Möbius inversion formula is equivalent to

$$p_\Psi(n) = \sum_{d\mid n} \mu(d) p(n/d).$$

Similarly for (1.3),

$$p(n,k) = \sum_{d\mid n} p_\Psi(n/d,k),$$

or equivalently

$$p_\Psi(n,k) = \sum_{d\mid n} \mu(d) p(n/d,k).$$

□

In particular, we find

$$p_\Psi(n,2) = \sum_{d\mid n} \mu(d) \left\lfloor \frac{n}{2d} \right\rfloor = \frac{1}{2}\phi(n),$$

where $\phi(n)$ is the Euler's totient function.

2. The results

Theorem 2.1.

(1) *If $n = 1, 2$, then $p(n,3) = 0$.*

(2) *If n is even, then*

$$p(n,3) \equiv 0 \pmod 2 \text{ if and only if } n^2 \equiv 0 \text{ or } 4 \pmod{24}$$

(3) *If n is odd such that $n > 2$, then*

$$p(n,3) \equiv 0 \pmod 2 \text{ if and only if } 3 \nmid n.$$

Proof. (1): The first statement is trivial.

(2): Assume that n is even. Using the identity $p_\Psi(n,3) = \langle n^2/12 \rangle$ we find that the case $(n/2)^2 \equiv 0 \pmod 3$ gives $p_\Psi(n,3) = n^2/12$ and the case $(n/2)^2 \equiv 1 \pmod 3$ yields $p_\Psi(n,3) = (n^2-4)/12$. Then in the former case we have

$$p_\Psi(n,3) \equiv 0 \pmod 2 \text{ if and only if } n^2 \equiv 0 \pmod{24}$$

and in the latter case we find

$$p_\Psi(n, 3) \equiv 0 \pmod 2 \text{ if and only if } n^2 \equiv 4 \pmod{24}.$$

(3): Assume now that $n > 2$ is odd and let $X \subseteq \mathbb{N}^3$ be defined as follows:

$$X = \{(x, y, z) : \ x \geq y \geq z, \ x \neq z, \text{ and } x + y + z = n\}.$$

Then

$$p(n, 3) = \begin{cases} |X \cup \{(n/3, n/3, n/3)\}| & \text{if } 3 \mid n, \\ |X| & \text{otherwise.} \end{cases}$$

Clearly it will be enough to show that $|X|$ is even. Note that if n is the sum of exactly three parts, then all of the parts are odd or exactly two of the parts are even. We now define a map $f : X \to X$ as follows.
If x, y, and z are all odd, then

$$f\big((x, y, z)\big) = \begin{cases} (x, y - 1, z + 1) & \text{if } x = y, \\ (x - 1, y + 1, z) & \text{otherwise.} \end{cases}$$

If y and z are even, then

$$f\big((x, y, z)\big) = \begin{cases} (x, y + 1, z - 1) & \text{if } x = y + 1, \\ (x - 1, y + 1, z) & \text{otherwise.} \end{cases}$$

If x and y are even, then

$$f\big((x, y, z)\big) = (x + 1, y - 1, z).$$

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$$f\big((x, y, z)\big) = (x + 1, y - 1, z).$$

Then it is easy to check that f is one-to-one, $f^{-1} = f$, and f fixes none of the elements of X. Thus X must have an even number of elements. This completes the proof. □

Clearly if $n = 1, 2$, then $p_\Psi(n, 3) = 0$ and if $n = 3, 4$, then $p_\Psi(n, 3) = 1$. As to $n > 4$ we have the following congruence.

Theorem 2.2. *If $n > 4$, then*

$$p_\Psi(n, 3) \equiv 0 \pmod 2.$$

Proof. Let $n > 4$. Assume first that n is even. Clearly if n is the sum of three relatively prime parts, then exactly one of these parts must be even. Let

$$P_e = \{\{2a, b, c\} : \ 2a + b + c = n \text{ and } \gcd\{2a, b, c\} = 1\}.$$

Then it is enough to prove that $|P_e|$ is even for even $n > 4$. We define a map $g : P_e \to P_e$ as follows.
If $b = c$, then

$$g(\{2a, b, b\}) = \begin{cases} \{2b, a, a\} & \text{if } a \text{ is odd,} \\ \{a, a + b, b\} & \text{if } a \text{ is even.} \end{cases}$$

If $b \neq c$, say $b > c$, then

$$g(\{2a, b, c\}) = \begin{cases} \{b-c, 2a+c, c\} & \text{if } b-c \neq 2a, \\ \{4a, c, c\} & \text{if } b-c = 2a. \end{cases}$$

Then it is easy to check that g is one-to-one, $g^{-1} = g$, and g fixes none of the elements of P_e. Then P_e must have an even number of elements. This proves the theorem if n is even and $n > 4$.

If n is odd such that $3 \nmid n$, then by Theorem 2.1 we have $p(n/d, 3) \equiv 0 \pmod 2$ for any $d \mid n$. It follows by Theorem 1.1 that

$$p_\Psi(n, 3) = \sum_{d|n} \mu(d) p(n/d, 3) \equiv 0 \pmod 2.$$

Assume now that n is an odd multiple of 3, say $n = 3^l m$ for positive integers l and m such that $6 \nmid m$. If $l = 1$ and $m > 1$, then by Theorems 1.1 and 2.1 we find

$$\begin{aligned} p_\Psi(3m, 3) &= \mu(3)p(m, 3) + \mu(3m)p(1, 3) + \sum_{d|m} \mu(d)p(3m/d, 3) \\ &\equiv 0 + 0 + \sum_{d|m} \mu(d) \pmod 2 \\ &\equiv 0 \pmod 2. \end{aligned}$$

If $l > 1$ and $m = 1$, then by Theorems 1.1 and 2.1 we have

$$p_\Psi(n, 3) = \mu(1)p(3^l, 3) + \mu(3)p(3^{l-1}, 3) \equiv 1 + 1 \pmod 2 \equiv 0 \pmod 2.$$

If $l > 1$ and $m > 1$, then again Theorems 1.1 and 2.1 yield

$$\begin{aligned} p_\Psi(3^l m, 3) &= \mu(3)p(3^{l-1}m, 3) + \mu(3m)p(3^{l-1}, 3) + \sum_{d|m} \mu(d)p(3^l m/d, 3) \\ &\equiv 1 + 1 + \sum_{d|m} \mu(d) \pmod 2 \\ &\equiv \sum_{d|m} \mu(d) \pmod 2 \\ &\equiv 0 \pmod 2. \end{aligned}$$

□

Remark: Using arguments of the present paper, other identities on partitions can be extended to corresponding relatively prime partitions. Examples include the identities for $s(n)$, $s_d(n)$, and $b_r(n)$ obtained in [3]. These are being looked at in a separate article.

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