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COEFFICIENTS OF CHROMATIC POLYNOMIALS AND TENSION POLYNOMIALS

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Abstract. We evaluate coefficients of the chromatic polynomial of a graph G as sums of zero values of tension polynomials of certain "maximal" subgraphs of G.

The chromatic polynomial $\chi_G(k)$ of a graph G evaluates the number of k-colorings of G. It is known that

$$\chi_G(k) = k^{c(G)} \cdot T_G(k), \tag{1}$$

where $T_G(k)$ is the tension polynomial of G and $c(G)$ is the number of components of G. For more details about the interpretation of $T_G(k)$, we refer to [1, 3, 5, 7]. Coefficients of chromatic polynomials are studied in [2, 6, 7]. In this paper we evaluate these coefficients using zero values of some tension polynomials.

If G is a graph, then $V(G)$ and $E(G)$ denote the vertex and edge sets of G, respectively. If $e \in E(G)$, then $G-e$ and G/e denote the graphs obtained from G after deleting and contracting e (i.e., deleting e and identifying its ends into a new vertex), respectively.

It is well known that (see, e.g., [5])

(2) $T_G(k) = 0$ if G has a loop,

(3) $T_G(k) = 1$ if $E(G) = \varnothing$,

(4) $T_G(k) = (k-1) \cdot T_{G-e}(k)$ if e is a bridge (1-edge cut) of G,

(5) $T_G(k) = T_{G-e}(k) - T_{G/e}(k)$ if e is not a bridge of G.

If G is a disjoint union of H_1, H_2, and G' is obtained from G after identifying a vertex from H_1 with a vertex from H_2, then (see [5])

$$T_G(k) = T_{G'}(k) = T_{H_1}(k) \cdot T_{H_2}(k). \tag{6}$$

By (3)–(5) and induction on $|E(G)|$ we can check that,

(7) $T_G(0)$ is a nonzero integer with sign $(-1)^{|V(G)|-c(G)}$

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for each graph G without loops. Items (1) and (7) indicate that $T_G(k)$ is a nontrivial divisor of $\chi_G(k)$.

If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X (i.e., $V(G[X]) = X$ and $E(G[X])$ consists of the edges of G with both ends from X). If $P = \{X_1, \dots, X_r\}$ is a partition of $V(G)$, then denote by $G[P]$ the disjoint union of $G[X_i]$, $i = 1, \dots, r$. Note that $|P| = r$. Denote by $\mathcal{P}_G$ the set of partitions of $V(G)$ such that $c(G[P]) = |P|$, (i.e., $P = \{X_1, \dots, X_r\} \in \mathcal{P}_G$ if and only if $G[X_i]$ is connected for every $i = 1, \dots, r$).

Theorem 1. *For every graph G,*

$$\chi_G(k) = \sum_{P \in \mathcal{P}_G} T_{G[P]}(0) \cdot k^{|P|}.$$

Proof. We use induction on $|E(G)|$. By (1)–(3), the statement holds true if $E(G) = \varnothing$ or G has a loop. Consider $e \in E(G)$ having two different ends u and v. It is well known (see [1, 7]) that

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k). \tag{8}$$

$\mathcal{P}_G$ ($\mathcal{P}_{G-e}$) is the disjoint union of $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ ($\mathcal{P}'_1$, $\mathcal{P}'_2$), where

$$\mathcal{P}_1 = \{P \in \mathcal{P}_G : e \in E(G[P]) \text{ is not a bridge of } G[P]\},$$
$$\mathcal{P}_2 = \{P \in \mathcal{P}_G : e \text{ is a bridge of } G[P]\},$$
$$\mathcal{P}_3 = \{P \in \mathcal{P}_G : e \notin E(G[P])\},$$
$$\mathcal{P}'_1 = \{P \in \mathcal{P}_{G-e} : u, v \text{ are in one component of } G[P]\},$$
$$\mathcal{P}'_2 = \{P \in \mathcal{P}_{G-e} : u, v \text{ are in two components of } G[P]\}.$$

Let w be the vertex of G/e arising from u and v after contracting e. If H is a subgraph of G/e containing w, then denote by $\rho(H)$ the subgraph of $G - e$ with vertex set $(V(H) \setminus w) \cup \{u, v\}$ and edge set $E(H)$ (supposing the ends of edges are the same as in $G - e$). Define

$$\mathcal{P}''_1 = \{P \in \mathcal{P}_{G/e} : u, v \text{ are in one component of } \rho(G[P])\},$$
$$\mathcal{P}''_2 = \{P \in \mathcal{P}_{G/e} : u, v \text{ are in two components of } \rho(G[P])\}.$$

$\mathcal{P}_{G/e}$ is the disjoint union of $\mathcal{P}''_1$, $\mathcal{P}''_2$.

$P \in \mathcal{P}_1$ if and only if $P \in \mathcal{P}'_1$, and if and only if the partition arising from P after identifying u and v into w belongs to $\mathcal{P}''_1$. Thus by (5),

$$\sum_{P \in \mathcal{P}_1} T_{G[P]}(0) k^{|P|} = \sum_{P \in \mathcal{P}'_1} T_{(G-e)[P]}(0) k^{|P|} - \sum_{P \in \mathcal{P}''_1} T_{(G/e)[P]}(0) k^{|P|}.$$

$P \in \mathcal{P}_2$ if and only if the partition arising from P after identifying u and v into w belongs to $\mathcal{P}''_2$ (note that $\mathcal{P}_2 = \mathcal{P}''_2 = \varnothing$ if e has a parallel edge). Thus by (4) and (6),

$$\sum_{P \in \mathcal{P}_2} T_{G[P]}(0) k^{|P|} = - \sum_{P \in \mathcal{P}''_2} T_{(G/e)[P]}(0) k^{|P|}.$$

$P \in \mathcal{P}_3$ if and only if $P \in \mathcal{P}'_2$, whence

$$\sum_{P \in \mathcal{P}_3} T_{G[P]}(0)k^{|P|} = \sum_{P \in \mathcal{P}'_2} T_{(G-e)[P]}(0)k^{|P|}.$$

Therefore

$$\sum_{P \in \mathcal{P}_G} T_{G[P]}(0)k^{|P|} = \sum_{P \in \mathcal{P}_{G-e}} T_{(G-e)[P]}(0)k^{|P|} - \sum_{P \in \mathcal{P}_{G/e}} T_{(G/e)[P]}(0)k^{|P|}.$$

$|E(G-e)|, |E(G/e)| < |E(G)|$, whence by the induction hypothesis,

$$\sum_{P \in \mathcal{P}_G} T_{G[P]}(0)k^{|P|} = \chi_{G-e}(k) - \chi_{G/e}(k).$$

and by (8),

$$\sum_{P \in \mathcal{P}_G} T_{G[P]}(0)k^{|P|} = \chi_G(k).$$

□

Denote by $\mathcal{P}_{G,r} = \{P \in \mathcal{P}_G; |P| = r\}$, $1 \leq r \leq |V(G)|$.

Theorem 2. *If G is a graph with n vertices and $\chi_G(k) = \sum_{r=0}^{n} \alpha_r \cdot k^r$, then*

$$\alpha_r = \sum_{P \in \mathcal{P}_{G,r}} T_{G[P]}(0) \text{ for } r=0,\dots,n.$$

Proof. This follows immediately from Theorem 1 and the definition of $\mathcal{P}_{G,r}$. □

Notice that $\alpha_r = 0$ and $\mathcal{P}_{G,r} = \varnothing$ for each $0 \leq r < c(G)$. Thus $\alpha_r = 0$ for each $0 \leq r < c(G)$, and the statement of Theorem 2 is nontrivial only for $r = c(G), \dots, n$.

If G has no loops, then by (7), $T_{G[P]}(0)$ has sign $(-1)^{n-r}$ for each $P \in \mathcal{P}_{G,r}$. Thus Theorem 2 gives a formula expressing α_r as a sum of numbers with the same sign. Hence α_r is a nonzero integer with sign $(-1)^{n-r}$ (see, e.g., [4, 7]).

Let us call a subgraph H of G *edge-maximal* if $V(H) = V(G)$ and each edge $e \in E(G) \setminus E(H)$ joins two components of H. Clearly, the set of graphs $G[P]$, $P \in \mathcal{P}_G$, equals the set of edge-maximal subgraphs of G. Thus by Theorem 2, $\alpha_r = \sum T_H(0)$ where the sum is considered over the set of edge-maximal subgraphs H of G with r components.

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