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A COMPLETE SPAN OF $\mathcal{H}(4,4)$ ADMITTING $PSL_2(11)$ AND RELATED STRUCTURES

ANTONIO COSSIDENTE, GARY L. EBERT, AND GIUSEPPE MARINO

Dedicated to the centenary of the birth of Ferenc Kárteszi (1907–1989).

Abstract. We construct a complete 11–span of $\mathcal{H}(4,4)$ admitting the group $\mathrm{PSL}_2(11)$. This span turns out to be associated with the unique Hadamard design $\mathcal{H}_{11}$ and the so-called Petersen design.

1. Introduction

A *spread* of a finite polar space $\mathcal{P}$ is a set of mutually skew subspaces of maximum dimension (*generators*) that cover $\mathcal{P}$. Any set of mutually skew generators is called a *partial spread* or *span*. If a spread does not exist, it is natural to ask how large a span can be. A span is said to be *complete* if it is maximal with respect to set–theoretic inclusion. An equally natural question to ask is the following: what is the smallest possible size for a complete span? In this paper the polar space involved is the Hermitian variety $\mathcal{H}(4,q^2)$ of $PG(4,q^2)$. It was shown in [1] that $\mathcal{H}(4,4)$ has no spread, but the existence of a spread of $\mathcal{H}(4,q^2)$, $q>2$, is still an open problem. On the other hand, in [7] it has been shown that a complete span of $\mathcal{H}(4,q^2)$ must have size at least $q^3+q\sqrt{q}-1/2q-3/8\sqrt{q}+7/8$. Hence, when $q=2$, a complete span of $\mathcal{H}(4,4)$ must have at least 11 lines.

Exploring the geometry of orbits of $\mathrm{PSL}_2(11)$ on generators of $\mathcal{H}(4,4)$, we construct an example of a complete 11–span of $\mathcal{H}(4,4)$ admitting the linear group $\mathrm{PSL}_2(11)$. This 11–span is associated with another complete 11–span (its companion), covering the same point set. We found that these two 11–spans give rise to the unique Hadamard design $\mathcal{H}_{11}$ in a very natural way. We also show that the Petersen design can be constructed from the orbits of $\mathrm{PSL}_2(11)$ on generators.

2. On the group $\mathrm{PSL}_2(11)$

From [9] one sees that $G=\mathrm{PSL}_2(11)$ has a 5-dimensional representation over $K=\mathrm{GF}(4)$ in which G fixes a non–degenerate Hermitian form on the

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underlying vector space V. Hence we see that $G < \mathrm{PSU}_5(4)$. It is this embedding that we wish to exploit in the following constructions.

Using MAGMA [3], we first construct the unique subgroup G of order 660 in $\mathrm{PSU}_5(4)$, where necessarily $G \cong \mathrm{PSL}_2(11)$ as above. The equation used by MAGMA for the invariant Hermitian variety $\mathcal{H}(4,4)$ is $x_0x_4^2+x_1x_3^2+x_2^3+x_3x_1^2+x_4x_0^2=0$. The group G has five orbits on the generators of $\mathcal{H}(4,4)$, say $L_1,\ldots,L_5$, with $|L_1|=|L_2|=11$, $|L_3|=|L_4|=55$, and $|L_5|=165$. Computer computations show that the generators of L_1 form an 11–span of $\mathcal{H}(4,4)$, as do the generators of L_2. These two spans cover the same set of 55 points on $\mathcal{H}(4,4)$, and any other generator meets this set in at least one point. Hence L_1 and L_2 are (companion) complete spans of $\mathcal{H}(4,4)$, meeting the lower bound mentioned above. Letting ω denote a primitive element of $K=\mathrm{GF}(4)$, these complete spans are the following:

$$
\begin{array}{lllllll}
L_1 = \{\{ & (1,\omega,\omega,0,\omega^2), & (1,\omega^2,\omega,1,0), & (1,1,\omega,\omega^2,1), & (0,1,0,1,\omega^2), & (1,0,\omega,\omega,\omega) & \}, \\
\{ & (1,1,\omega^2,\omega,1), & (0,1,0,1,\omega), & (1,\omega,\omega^2,1,0), & (1,0,\omega^2,\omega^2,\omega^2), & (1,\omega^2,\omega^2,0,\omega) & \}, \\
\{ & (0,1,\omega^2,\omega^2,1), & (1,1,\omega,0,\omega), & (1,0,1,\omega^2,\omega^2), & (1,\omega^2,\omega^2,1,0), & (1,\omega,0,\omega,1) & \}, \\
\{ & (1,\omega^2,\omega^2,\omega^2,\omega^2), & (0,1,\omega^2,\omega^2,\omega), & (1,1,\omega,\omega,0), & (1,\omega,0,0,1), & (1,0,1,1,\omega) & \}, \\
\{ & (1,\omega,\omega,1,1), & (1,\omega^2,1,\omega^2,\omega^2), & (1,0,\omega^2,\omega,\omega), & (0,1,\omega^2,\omega,\omega), & (1,1,0,0,0) & \}, \\
\{ & (1,1,0,\omega^2,\omega), & (1,\omega^2,\omega^2,\omega,0), & (1,\omega,1,1,1), & (0,1,\omega,\omega^2,1), & (1,0,\omega,0,\omega^2) & \}, \\
\{ & (0,1,\omega^2,\omega,1), & (1,\omega^2,\omega,0,\omega^2), & (1,0,0,1,0), & (1,\omega,1,\omega,\omega), & (1,1,\omega^2,\omega^2,1) & \}, \\
\{ & (1,0,\omega,1,\omega), & (1,1,1,\omega,1), & (1,\omega,\omega^2,0,\omega^2), & (0,1,\omega^2,\omega^2,\omega^2), & (1,\omega^2,0,\omega^2,0) & \}, \\
\{ & (1,1,1,0,\omega), & (1,\omega^2,0,1,\omega), & (1,\omega,\omega^2,\omega,\omega), & (1,0,\omega,\omega^2,\omega), & (0,1,\omega^2,\omega^2,0) & \}, \\
\{ & (0,1,\omega,\omega,\omega), & (0,1,\omega,\omega,\omega^2), & (0,0,0,0,1), & (0,1,\omega,\omega,0), & (0,1,\omega,\omega,1) & \}, \\
\{ & (1,\omega^2,1,0,\omega^2), & (1,1,\omega,1,\omega), & (1,\omega,\omega^2,\omega^2,0), & (0,1,\omega,\omega^2,\omega^2), & (1,0,0,\omega,1) & \}\}, \\
L_2 = \{\{ & (1,0,\omega^2,\omega,\omega), & (0,1,0,1,\omega^2), & (1,\omega,\omega^2,0,\omega^2), & (1,\omega^2,\omega^2,1,0), & (1,1,\omega^2,\omega^2,1) & \}, \\
\{ & (0,1,\omega^2,\omega^2,1), & (0,1,\omega^2,\omega^2,\omega), & (0,0,0,0,1), & (0,1,\omega^2,\omega^2,\omega^2), & (0,1,\omega^2,\omega^2,0) & \}, \\
\{ & (1,\omega^2,1,\omega^2,\omega^2), & (1,\omega,\omega^2,1,0), & (0,1,\omega,\omega,\omega^2), & (1,1,\omega,0,\omega), & (1,0,0,\omega,1) & \}, \\
\{ & (1,\omega^2,\omega^2,\omega^2,\omega^2), & (0,1,\omega,\omega,\omega), & (1,\omega,1,1,1), & (1,0,\omega,\omega,\omega), & (1,1,0,0,0) & \}, \\
\{ & (1,1,\omega^2,\omega,1), & (1,\omega,\omega,0,\omega^2), & (0,1,\omega,\omega^2,\omega^2), & (1,0,1,1,\omega), & (1,\omega^2,0,\omega^2,0) & \}, \\
\{ & (1,\omega,\omega,1,1), & (0,1,0,1,\omega), & (1,1,\omega,\omega,0), & (1,\omega^2,\omega,0,\omega^2), & (1,0,\omega,\omega^2,\omega) & \}, \\
\{ & (1,\omega,\omega^2,\omega^2,0), & (1,1,1,\omega,1), & (0,1,\omega^2,\omega,\omega), & (1,\omega^2,0,1,\omega), & (1,0,\omega,0,\omega^2) & \}, \\
\{ & (1,\omega^2,1,0,\omega^2), & (1,1,\omega,\omega^2,1), & (1,0,0,1,0), & (1,\omega,\omega^2,\omega,\omega), & (0,1,\omega,\omega,1) & \}, \\
\{ & (1,0,\omega,1,\omega), & (1,1,0,\omega^2,\omega), & (0,1,\omega,\omega,0), & (1,\omega^2,\omega^2,0,\omega), & (1,\omega,1,\omega,\omega) & \}, \\
\{ & (1,1,\omega,1,\omega), & (0,1,\omega^2,\omega,1), & (1,\omega,0,0,1), & (1,0,1,\omega^2,\omega^2), & (1,\omega^2,\omega^2,\omega,0) & \}, \\
\{ & (1,1,1,0,\omega), & (1,\omega^2,\omega,1,0), & (1,0,\omega^2,\omega^2,\omega^2), & (0,1,\omega,\omega^2,1), & (1,\omega,0,\omega,1) & \}\}.
\end{array}
$$

Now every line in L_2 meets exactly 5 lines of L_1 (in one point each), and hence we determine 11 (distinct) subsets of L_1, each of size 5. Moreover, direct computations show that each pair of lines in L_1 lies in exactly 2 of these subsets. That is, we have constructed the $2-(11,5,2)$ biplane. This, of course, is the famous Hadamard design $\mathcal{H}_{11}$. The complementary design is a $2-(11,6,3)$ BIBD.

One of the G-orbits on generators of size 55, say L_4, has the property that each line of L_4 meets exactly 3 lines of L_1, and hence we so determine 55 distinct subsets of L_1, each of size 3. More direct computations show that each pair of lines in L_1 lies in exactly 3 such subsets, and hence we have a $2-(11,3,3)$ BIBD. This design is known as the *Petersen design* [2].

From a group–theoretic point of view, the Petersen design can be described as follows. There are 55 involutions in $\mathrm{PSL}_2(11)$, and each of them has as axis a generator in L_4 [4], and each of them fixes 3 lines in L_1.

We also make the following group-theoretic observation.

Proposition 2.1. *G is a maximal subgroup of* $\mathrm{PSU}_5(4)$.

Proof. The group G has just two orbits on points of $\mathcal{H}(4,4)$, one of size 55 and one of size 110. Let F be any subgroup of $\mathrm{PSU}_5(4)$ such that

$$G < F \leq \mathrm{PSU}_5(4).$$

Since G is the full stabilizer of L_1 in $\mathrm{PSU}_5(4)$, F must act transitively on points of $\mathcal{H}(4,4)$. From [6, Cor. 5.12] necessarily $F = \mathrm{PSU}_5(4)$, and hence G is maximal in $\mathrm{PSU}_5(4)$. $\square$

3. $\mathrm{PSL}_2(11)$ as a subgroup of $\mathrm{PSU}_6(4)$

Embedding $\mathcal{H}(4,4)$ in $\mathcal{H}(5,4)$, we now look at the action of $\mathrm{PSL}_2(11)$ on generators of $\mathcal{H}(5,4)$. We again use MAGMA to find the unique subgroup G of order 660 in $\mathrm{PSU}_6(4)$, where necessarily $G \cong \mathrm{PSL}_2(11)$. The equation used by MAGMA for $\mathcal{H}(5,4)$ is $x_0x_5^2+x_1x_4^2+x_2x_3^2+x_3x_2^2+x_4x_1^2+x_5x_0^2=0$. Computer computations show that G has 15 orbits on the generators of $\mathcal{H}(5,4)$, of which 6 have size 11. The 11 planes in any such orbit have the property that any two of them meet in exactly one point. Moreover, we can pair off these 6 orbits in such a way that, upon taking the union of each pair of orbits, we obtain 3 sets of 22 generators of $\mathcal{H}(5,4)$ with this same intersection property. That is, any two planes from the same set of 22 generators will meet in exactly one point. It is easy to check that these three sets form three distinct 2-dimensional dual hyperovals embedded in $\mathcal{H}(5,4)$. We recall that, a family F of 2-dimensional subspaces of the finite 5-dimensional projective space $PG(5,4)$ is called a dual hyperoval if: every point of $PG(5,4)$ belongs to either 0 or 2 members in F; any two members of F have exactly one point in common; and if the set of points belonging to the member of F spans $PG(5,4)$. It turns out that the three dual hyperovals admit the Mathieu group M_{22} inside their groups of automorphisms [5].

Let $\{P_1, P_4\}$ be such a pair of orbits, and let $P = P_1 \cup P_4$ denote the union of these orbits. Thus P consists of 22 generators of $\mathcal{H}(5,4)$, any two of which meet in exactly one point.

The number of points covered by the generators in P_1 is 176, as is true for the generators in P_4. The intersection $\mathcal{Q}$ of these two point sets is a collection of 121 points on $\mathcal{H}(5,4)$ with the property that each such point lies on exactly one generator from the orbit P_1 and lies on exactly one generator from the orbit P_4. Thus we get a type of "grid" induced on this set of points in $\mathcal{H}(5,4)$.

Now let π_1 denote some plane in P_1. MAGMA computations show that the stabilizer H_1 of π_1 in G is isomorphic to the alternating group A_5, and H_1 has three point orbits on π_1. These point orbits have sizes 5, 6, and 10, with the point orbit O_1 of size 6 being a hyperoval in π_1. More direct computations show that the orbit of O_1 under G yields a collection of 11 disjoint hyperovals covering 66 points of $\mathcal{Q}$. Each plane of P_1 contains one of these hyperovals. Let $\mathcal{O}_1$ denote this collection of disjoint hyperovals. Similarly, starting with some plane π_4 in P_4, one obtains another collection $\mathcal{O}_4$ of 11 disjoint hyperovals, one contained in each plane of P_4. The hyperoval collection is the following:

$$
\begin{array}{rlllll}
\mathcal{O}_1 = \{\{ & (0,0,1,0,\omega,1), & (1,0,1,1,1,1), & (0,1,0,\omega^2,1,\omega), & \\
& (1,1,\omega,\omega,1,0), & (1,\omega^2,0,\omega^2,0,1), & (1,\omega,\omega^2,0,0,0) & \}, \\
\{ & (1,1,0,0,\omega,\omega), & (0,1,0,\omega,0,\omega^2), & (1,\omega^2,0,1,\omega,\omega^2), & \\
& (1,0,0,1,\omega,1), & (1,\omega,0,0,\omega,0), & (0,0,0,1,0,0) & \}, \\
\{ & (1,1,1,1,0,0), & (0,1,0,0,0,0), & (1,\omega^2,1,\omega^2,0,\omega), & \\
& (0,0,0,1,0,1), & (1,\omega,1,0,0,1), & (1,0,1,\omega,0,\omega^2) & \}, \\
\{ & (0,1,\omega,1,\omega^2,0), & (1,0,0,0,0,1), & (1,\omega^2,1,\omega,1,1), & \\
& (0,0,0,1,\omega^2,0), & (1,\omega,\omega^2,0,0,1), & (1,1,\omega,\omega,1,1) & \}, \\
\{ & (1,\omega^2,1,1,\omega,\omega^2), & (0,0,0,0,,\omega), & (1,\omega^2,\omega^2,\omega^2,0,0), & \\
& (0,0,1,1,0,0), & (1,\omega^2,\omega,\omega,\omega^2,1), & (1,\omega^2,0,0,1,\omega) & \}, \\
\{ & (0,1,0,\omega^2,0,1), & (1,\omega,\omega,1,\omega,\omega), & (1,\omega^2,\omega,1,\omega,1), & \\
& (1,0,\omega,\omega,\omega,1), & (1,1,\omega,\omega,\omega,\omega), & (0,0,0,1,0,\omega^2) & \}, \\
\{ & (1,\omega^2,0,\omega^2,\omega^2,1), & (1,1,\omega,\omega,0,0), & (0,1,1,\omega,\omega^2,1), & \\
& (1,\omega,1,\omega,\omega^2,1), & (1,0,\omega^2,\omega^2,0,0), & (0,0,0,1,1,\omega) & \}, \\
\{ & (1,\omega^2,1,\omega^2,\omega^2,\omega) & (1,0,\omega,\omega^2,0,\omega^2), & (0,1,\omega,0,1,\omega^2), & \\
& (0,1,0,0,1,0), & (1,0,\omega^2,\omega^2,0,1), & (1,\omega^2,0,\omega^2,\omega^2,0) & \}, \\
\{ & (1,1,1,\omega,1,\omega^2), & (1,\omega,\omega,0,0,1), & (1,1,\omega,1,\omega^2,0), & \\
& (1,\omega^2,0,0,0,1), & (1,\omega^2,1,1,\omega^2,0), & (1,\omega,0,\omega,1,\omega^2) & \}, \\
\{ & (1,\omega^2,\omega,1,0,\omega), & (1,\omega^2,\omega,\omega^2,\omega^2,\omega), & (1,\omega,1,0,\omega^2,\omega^2), & \\
& (1,\omega,1,1,1,\omega^2), & (1,0,0,\omega^2,0,1), & (1,0,0,0,1,1) & \}, \\
\{ & (1,\omega^2,1,1,\omega,\omega), & (1,\omega^2,\omega^2,\omega^2,0,1), & (1,1,0,\omega,1,0), & \\
& (1,1,\omega,0,\omega^2,\omega^2), & (0,1,1,0,0,\omega), & (0,1,0,1,1,0) & \}\}.
\end{array}
$$

$$
\begin{aligned}
\mathcal{O}_4 = \{\{ & (1,1,\omega,\omega,1,1), & (1,0,\omega^2,\omega^2,0,1), & (1,\omega,1,0,\omega^2,\omega^2), \\
& (1,\omega^2,0,1,\omega,\omega^2), & (0,1,1,0,0,\omega), & (0,0,0,1,1,\omega) \quad \}, \\
\{ & (1,\omega^2,1,1,\omega,\omega), & (1,\omega,\omega,1,\omega,\omega), & (1,\omega^2,0,0,0,1), \\
& (0,1,0,\omega^2,1,\omega), & (1,\omega,\omega^2,0,0,1), & (0,1,1,\omega,\omega^2,1) \quad \}, \\
\{ & (1,\omega,\omega,0,0,1), & (1,\omega^2,\omega,1,0,\omega), & (0,1,0,0,0,0), \\
& (1,0,\omega,\omega^2,0,\omega^2), & (1,1,\omega,\omega,0,0), & (0,0,0,1,0,\omega^2) \quad \}, \\
\{ & (1,\omega^2,1,\omega^2,\omega^2,\omega), & (1,1,0,0,\omega,\omega), & (1,\omega^2,0,\omega^2,0,1), \\
& (1,0,\omega,\omega,\omega,1), & (1,1,\omega,0,\omega^2,\omega^2), & (1,0,1,\omega,0,\omega^2) \quad \}, \\
\{ & (1,\omega^2,\omega,1,\omega,1), & (1,\omega^2,\omega,\omega,\omega^2,1), & (1,\omega,1,0,0,1), \\
& (1,0,0,1,\omega,1), & (1,\omega,1,\omega,\omega^2,1), & (1,0,0,0,0,1) \quad \}, \\
\{ & (1,\omega^2,1,1,\omega,\omega^2), & (0,1,\omega,0,1,\omega^2), & (0,0,0,1,\omega^2,0), \\
& (1,1,\omega,\omega,\omega,\omega), & (1,\omega,\omega^2,0,0,0), & (1,0,0,\omega^2,0,1) \quad \}, \\
\{ & (0,1,0,0,1,0), & (0,0,1,1,0,0), & (1,1,\omega,\omega,1,0), \\
& (1,\omega^2,1,1,\omega^2,0), & (1,\omega,0,0,\omega,0), & (1,0,\omega^2,\omega^2,0,0) \quad \}, \\
\{ & (1,1,1,\omega,1,\omega^2), & (1,0,1,1,1,1), & (0,1,0,\omega,0,\omega^2), \\
& (1,\omega,1,1,1,\omega^2), & (1,\omega^2,1,\omega,1,1), & (0,0,0,1,0,1) \quad \}, \\
\{ & (0,1,0,\omega^2,0,1), & (1,1,0,\omega,1,0), & (1,\omega^2,0,0,1,\omega), \\
& (1,\omega,0,\omega,1,\omega^2), & (0,0,0,1,0,0), & (1,0,0,0,1,1) \quad \}, \\
\{ & (1,1,1,1,0,0), & (1,1,\omega,1,\omega^2,0), & (1,\omega^2,\omega^2,\omega^2,0,0), \\
& (1,\omega^2,0,\omega^2,\omega^2,0), & (0,1,\omega,1,\omega^2,0), & (0,1,0,1,1,0) \quad \}, \\
\{ & (1,\omega^2,0,\omega^2,\omega^2,1), & (1,\omega^2,\omega^2,\omega^2,0,1), & (0,0,1,0,\omega,1), \\
& (0,0,0,0,1,\omega), & (1,\omega^2,\omega,\omega^2,\omega^2,\omega), & (1,\omega^2,1,\omega^2,0,\omega) \quad \}\}.
\end{aligned}
$$

The set of 66 points covered by the hyperovals in $\mathcal{O}_1$ is exactly the same subset of $\mathcal{Q}$ as that covered by the hyperovals in $\mathcal{O}_1$. Call this point set $\mathcal{Q}_0$. From the construction it follows that every hyperoval belonging to $\mathcal{O}_1$ meets every hyperoval belonging to $\mathcal{O}_4$ in at most one point. Thus each hyperoval in $\mathcal{O}_4$ meets exactly 6 hyperovals in $\mathcal{O}_1$, thereby determining 11 distinct subsets of $\mathcal{O}_1$, each of size 6. A direct MAGMA computation shows that each pair of hyperovals from $\mathcal{O}_1$ lies in exactly 3 such subsets, and hence we have a $2-(11,6,3)$ BIBD. The complementary design is a $2-(11,5,2)$ biplane, namely the Hadamard design $\mathcal{H}_{11}$.

One can also obtain the Hadamard design directly as follows. Choose some point $X \in \mathcal{Q}_0$, and take the point orbit R of X under a Sylow 11-subgroup of G. This orbit has size 11, and each hyperoval in either $\mathcal{O}_1$ or $\mathcal{O}_4$ contains exactly one point in R. The points in R are the following:

$$
\begin{aligned}
R \quad = \quad \{ \quad & (0,1,\omega,0,1,\omega^2), & (0,1,1,0,0,\omega), & (1,0,0,0,0,1), \\
& (1,\omega,0,0,\omega,0), & (1,0,1,\omega,0,\omega^2), & (0,0,0,0,1,\omega), \\
& (1,1,\omega,\omega,0,0), & (0,1,0,\omega^2,0,1), & (1,\omega,1,1,1,\omega^2), \\
& (1,1,\omega,1,\omega^2,0), & (0,1,0,\omega^2,1,\omega) & \quad \}.
\end{aligned}
$$

Thus removing the points of R yields two collections of 11 disjoint ovals (actually, conics), say $\mathcal{C}_1$ and $\mathcal{C}_4$, that cover the same set of 55 points in $\mathcal{H}(5,4)$. As above, each oval in $\mathcal{C}_4$ meets 5 ovals in $\mathcal{C}_1$ in one point each, thereby determining 11 distinct subsets of $\mathcal{C}_1$ of size 5. Direct computations show that any pair of ovals from $\mathcal{C}_1$ lies in exactly 2 such subsets. Hence we again have the design $\mathcal{H}_{11}$.

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Dipartimento di Matematica, Università degli Studi della Basilicata,
85100 Potenza, Italy
E-mail address: cossidente@unibas.it

Department of Mathematical Sciences, University of Delaware
Newark, DE 19716, U.S.A.
E-mail address: ebert@math.udel.edu

Dipartimento di Matematica e Appl., Università di Napoli “Federico II”,
80126 Napoli, Italy
E-mail address: giuseppe.marino@unina2.it