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PARTIALLY CRITICAL INDECOMPOSABLE GRAPHS

ANDREW BREINER, JITENDER DEOGUN, AND PIERRE ILLE

Abstract. Given a graph $G = (V, E)$, with each subset X of V is associated the subgraph $G(X)$ of G induced by X. A subset I of V is an interval of G provided that for any $a, b \in I$ and $x \in V \setminus I$, $\{a, x\} \in E$ if and only if $\{b, x\} \in E$. For example, $\varnothing$, $\{x\}$, where $x \in V$, and V are intervals of G called trivial intervals. A graph is indecomposable if all its intervals are trivial; otherwise, it is decomposable. Given an indecomposable graph $G = (V, E)$, consider a proper subset X of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. The graph G is critical according to $G(X)$ if for every $x \in V \setminus X$, $G(V \setminus \{x\})$ is decomposable. A graph is partially critical if it is critical according to one of its indecomposable subgraphs containing at least 4 vertices. In this paper, we characterize the partially critical graphs.

1. Definitions and notations

A *graph* G is defined by a finite and nonempty set V of *vertices* and by a family E of pairs of vertices called *edges*. Such a graph is denoted by (V, E). With each nonempty subset X of V associate the *subgraph* $G(X) = (X, \{\{x, y\} : \{x, y\} \in E; x, y \in X\})$ of G *induced* by X. For convenience, given $X \subseteq V$, $G(V \setminus X)$ is also denoted by $G - X$ and $G - \{x\}$ by $G - x$ for $x \in V$.

For instance, given a set V, $(V, \varnothing)$ is the *empty* graph on V whereas $(V, \{\{x, y\}; x \neq y \in V\})$ is the *complete* graph. Let $G = (V, E)$ be a graph, and consider a partition p of V. The graph G is *multipartite* by p if for every $M \in p$, $G(M)$ is empty. It is *bipartite* when $|p| = 2$.

Given graphs $G = (V, E)$ and $G' = (V', E')$, a bijection f from V onto V' is an *isomorphism* from G onto G' provided that for any $x, y \in V$, $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$. Two graphs are then said to be *isomorphic* if there exists an isomorphism from one onto the other. Given a graph G, an isomorphism from G onto itself is called an *automorphism* of G. The family of all the automorphisms of G constitutes a group, called the *automorphism group* of G and denoted by $\mathrm{Aut}(G)$.

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With each graph $G = (V, E)$ associate its *complement* $\overline{G} = (V, \overline{E})$ defined as follows: given $x \neq y \in V$, $\{x, y\} \in \overline{E}$ if $\{x, y\} \notin E$.

Given a graph $G = (V, E)$, we define an equivalence relation $\mathcal{C}$ on V in the following way. For any $x \neq y \in V$, $x\mathcal{C}y$ if there are vertices $x = x_0, \ldots, x_n = y$ such that $\{x_i, x_{i+1}\} \in E$ for $0 \leq i \leq n-1$. The equivalence classes of $\mathcal{C}$ are called the *connected components* of G. A vertex x of G is *isolated* if $\{x\}$ constitutes a connected component of G. The graph G is *connected* if V is the unique connected component of G. For example, given $n \geq 2$, the *path* on n vertices $P_n = (\{0, \ldots, n-1\}, \{\{i, j\} : \ |i-j| = 1\}_{i,j \in \{0,\ldots,n-1\}})$ is connected.

A *perfect matching* (called simply *matching* in this paper) of a graph $G = (V, E)$ is a subset of E which realizes a partition of V. Clearly, if a graph $G = (V, E)$ admits a matching, then $|V|$ is even. For instance, the family of pairs $\{\{2i, 2i+1\}; 0 \leq i \leq n/2 - 1\}$ is the unique matching of P_n when n is even.

Given a graph $G = (V, E)$, a subset I of V is an *interval* [4, 5] of G provided that for all $a, b \in I$ and $x \in V \setminus I$, $\{a, x\} \in E$ if and only if $\{b, x\} \in E$. For example, $\varnothing$, $\{x\}$, where $x \in V$, and V are intervals of G called *trivial intervals*. A graph is indecomposable [4, 5] if all its intervals are trivial; otherwise, it is decomposable. Given a graph G, since G and $\overline{G}$ share the same intervals, we obtain that G is indecomposable if and only if $\overline{G}$ is indecomposable. Note that the connected components of a graph are intervals, thus a non-connected graph with at least three vertices is decomposable. Lastly, we recall that a graph (V, E), with $|V| = 3$, is decomposable and that a graph (V, E), with $|V| = 4$, is indecomposable if and only if it is isomorphic to P_4. Furthermore, for every $n \geq 4$, P_n is indecomposable.

Given an indecomposable graph $G = (V, E)$, with $|V| \geq 2$, G is *critical* [5] if for each vertex x of G, $G - x$ is decomposable. For instance, the graph $G_{2n} = (\{0, \ldots, 2n-1\}, E_{2n})$, shown in Figure 1, is critical for $n \geq 2$, and is defined as follows. For $x, y \in \{0, \ldots, 2n-1\}$, $\{x, y\} \in E_{2n}$ if there exist $i \leq j \in \{0, \ldots, n-1\}$ such that $\{x, y\} = \{2i, 2j+1\}$.

Now, we introduce the following weakening of the criticality. Given an indecomposable graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. The graph G is called *critical according to* $G(X)$ (or X-critical [1]) if for every $x \in V \setminus X$, $G - x$ is decomposable. Moreover, a graph $G = (V, E)$ is said to be *partially critical* if there exists $X \subset V$ such that $|X| \geq 4$, $G(X)$ is indecomposable and G is critical according to $G(X)$.

In this paper, we characterize the class of partially critical graphs. The rest of the paper is organized as follows. In Section 2, we review relevant properties of indecomposable graphs. In Section 3, we give two examples of partially critical graphs. We introduce preliminary properties of partially critical graphs in Section 4, and in Section 5, we present our main results.

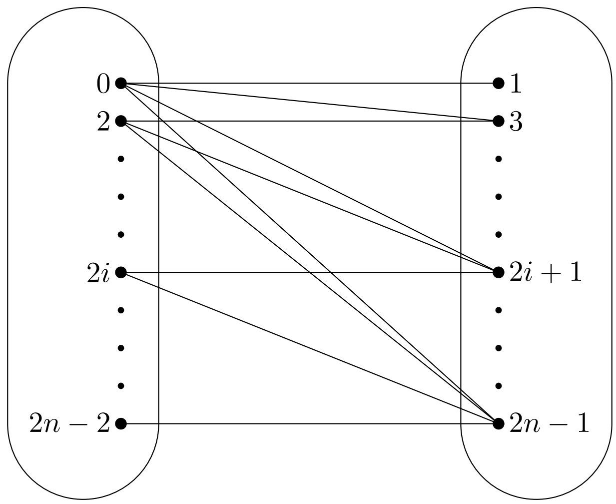

FIGURE 1. G_{2n}

2. Preleminaries

In this section, we review relevant properties of indecomposable graphs. We begin with the well known properties of the intervals of a graph. Then we examine the indecomposable subgraphs of an indecomposable graph. Finally, we recall the characterization of critical graphs [5].

Proposition 2.1. *Let $G = (V, E)$ be a graph.*

1. *Given a nonempty subset W of V, if I is an interval of G, then $I \cap W$ is an interval of $G(W)$.*
2. *If I and J are intervals of G, then $I \cap J$ is an interval of G.*
3. *If I and J are intervals of G such that $I \cap J \neq \varnothing$, then $I \cup J$ is an interval of G.*
4. *If I and J are intervals of G such that $I \setminus J \neq \varnothing$, then $J \setminus I$ is an interval of G.*

We examine the indecomposable subgraphs of an indecomposable graph in the next results.

Proposition 2.2 (Sumner [6])**.** *Given a graph $G = (V, E)$, with $|V| \geq 4$, if G is indecomposable, then there is a subset X of V such that $|X| = 4$, $G(X)$ is indecomposable and hence is isomorphic to P_4.*

To construct larger indecomposable subgraphs, we define the following partition from any indecomposable subgraph.

Given a graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. We consider the following subsets of $V \setminus X$.

- $\mathrm{Ext}(X)$ is the family of the elements x of $V \setminus X$ such that $G(X \cup \{x\})$ is indecomposable;

- $[X]$ is the family of the elements x of $V \setminus X$ such that X is an interval of $G(X \cup \{x\})$;
- For each $u \in X$, $X(u)$ is the family of the elements x of $V \setminus X$ such that $\{u, x\}$ is an interval of $G(X \cup \{x\})$.

The family $\{\mathrm{Ext}(X), [X]\} \cup \{X(u); u \in X\}$ is denoted by p_X. Furthermore, $[X]$ is divided into X^- and X^+ as follows.

- X^- is the set of the elements x of $V \setminus X$ such that for every $y \in X$, $\{x, y\} \notin E$;
- X^+ is the set of the elements x of $V \setminus X$ such that for every $y \in X$, $\{x, y\} \in E$.

Similarly, for each $u \in X$, $X(u)$ is divided into $X^-(u)$ and $X^+(u)$ as follows.

- $X^-(u)$ is the set of the elements x of $X(u)$ such that $\{u, x\} \notin E$;
- $X^+(u)$ is the set of the elements x of $X(u)$ such that $\{u, x\} \in E$.

We then introduce the three families below.

- $q_X = \{\mathrm{Ext}(X), X^-, X^+\} \cup \{X^-(u), X^+(u)\}_{u \in X}$;
- $q_X^- = \{X^-\} \cup \{X^-(u); u \in X\}$;
- $q_X^+ = \{X^+\} \cup \{X^+(u); u \in X\}$.

The family p_X is used as follows.

Lemma 2.3 (Ehrenfeucht and Rozenberg [2])**.** *Given a graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. The family p_X realizes a partition of $V \setminus X$. Moreover, the following assertions are satisfied.*

(1) *Given $x \neq y \in \mathrm{Ext}(X)$, if $G(X \cup \{x, y\})$ is decomposable, then $\{x, y\}$ is an interval of $G(X \cup \{x, y\})$.*
(2) *Given $x \in X(u)$ and $y \in V \setminus (X \cup X(u))$, where $u \in X$, if $G(X \cup \{x, y\})$ is decomposable, then $\{u, x\}$ is an interval of $G(X \cup \{x, y\})$.*
(3) *Given $x \in [X]$ and $y \in V \setminus (X \cup [X])$, if $G(X \cup \{x, y\})$ is decomposable, then $X \cup \{y\}$ is an interval of $G(X \cup \{x, y\})$.*

Corollary 2.4 (Ehrenfeucht and Rozenberg [2])**.** *Given a graph $G = (V, E)$, let X be a subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. If G is indecomposable and $|V \setminus X| \geq 2$, then there exist $x \neq y \in V \setminus X$ such that $G(X \cup \{x, y\})$ is indecomposable.*

Given a graph $G = (V, E)$, let X be a subset of V such that $|X| \geq 4$, $|V \setminus X| \geq 2$ and $G(X)$ is indecomposable. This corollary leads to the definition of the graph $G_X = (V \setminus X, E_X)$ in the following manner. For any $x \neq y \in V \setminus X$, $\{x, y\} \in E_X$ if $G(X \cup \{x, y\})$ is indecomposable.

Remark 2.5: Given a graph $G = (V, E)$, let X be a subset of V such that $|X| \geq 4$, $|V \setminus X| \geq 2$ and $G(X)$ is indecomposable. Consider distinct elements x and y of $V \setminus X$. If $x, y \in [X]$, then X is an interval of $G(X \cup \{x, y\})$. Given $u \in X$, if $x, y \in X(u)$, then $\{u, x, y\}$ is an interval of $G(X \cup \{x, y\})$.

Consequently, for each $M \in p_X \setminus \{\text{Ext}(X)\}$, $G_X(M)$ is empty. In other words, if $\text{Ext}(X) = \varnothing$, then G_X is multipartite by p_X.

Now, we study the intervals of G_X.

Lemma 2.6. *Given a graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. If $\text{Ext}(X) = \varnothing$, then the following two assertions hold.*

1. *If I is an interval of G such that $I \cap X = \varnothing$, then I is an interval of G_X which is included in an element of q_X.*
2. *Given $M \in p_X$ and $N \in q_X$ such that $N \subseteq M$, if I is an interval of G_X such that $I \subseteq N$ and if I is an interval of $G(M)$, then I is an interval of G.*

Proof. Let I be an interval of G such that $I \cap X = \varnothing$. Given $i \neq j \in I$, we have $I \cap (X \cup \{i, j\}) = \{i, j\}$ is an interval of $G(X \cup \{i, j\})$. Consequently, the function, defined on $X \cup \{i\}$ by $i \mapsto j$ and $k \mapsto k$ for $k \in X$, is an isomorphism from $G(X \cup \{i\})$ onto $G(X \cup \{j\})$. It follows from the definition of p_X and of q_X that i and j belong to the same element of q_X. Therefore, there exists $N \in q_X$ such that $I \subseteq N$. The unique element of p_X containing N is denoted by M. By Remark 2.5, $G_X(M)$ is empty and thus I is an interval of $G_X(M)$. Consequently, consider an element x of $(V \setminus X) \setminus M$. Firstly, assume that $M = X(u)$, where $u \in X$, and, for instance, that $\{u, x\} \in E$. By Lemma 2.3, for every $i \in I$, $\{i, x\} \in E$ if and only if $\{i, x\} \notin E_X$. Secondly, assume that $M = [X]$ and, for example, that $N = X^-$. It follows from Lemma 2.3 that for every $i \in I$, $\{i, x\} \in E$ if and only if $\{i, x\} \in E_X$. Thus the first assertion of the lemma follows.

Given $M \in p_X$ and $N \in q_X$, $N \subseteq M$, let I be an interval of G_X such that $I \subseteq N$ and I is an interval of $G(M)$. Consider an element x of $V \setminus M$. When $x \in (V \setminus X) \setminus M$, we proceed in a way similar to the case of first assertion. Consequently, assume that $x \in X$. If $M = [X]$, then either $I \subseteq X^-$ or $I \subseteq X^+$. In both cases, I is an interval of $G(I \cup \{x\})$. Lastly, assume that $M = X(u)$, where $u \in X$. As $\{u\} \cup X(u)$ is an interval of $G(X \cup X(u))$, I is an interval of $G(I \cup \{x\})$ when $x \neq u$. When $x = u$, it is sufficient to distinguish $N = X^-(u)$ and $N = X^+(u)$. $\square$

To continue, we examine the isolated vertices of G_X.

Lemma 2.7. *Given a graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable.*

1. *If I is an interval of G such that $X \subseteq I$, then the elements of $V \setminus I$ are isolated vertices of G_X.*
2. *Given $u \in X$, if I is an interval of G such that $I \cap X = \{u\}$, then the elements of $I \setminus \{u\}$ are isolated vertices of G_X.*

Consequently, if G admits a non trivial interval I such that $I \cap X \neq \varnothing$, then G_X possesses isolated vertices.

Proof. Firstly, let I be an interval of G such that $X \subseteq I$. We have $V \setminus I \subseteq [X]$. Let x be an element of $V \setminus I$. For every $y \in [X]$, X is an interval of $G(X \cup \{x, y\})$. Furthermore, for every $y \in (V \setminus X) \setminus [X]$, $y \in I$ and hence $I \cap (X \cup \{x, y\}) = X \cup \{y\}$ is an interval of $G(X \cup \{x, y\})$. Therefore, $\{x, y\} \notin E_X$ for every $y \in (V \setminus X) \setminus \{x\}$.

Secondly, given $u \in X$, let I be an interval of G such that $I \cap X = \{u\}$. We have $I \setminus \{u\} \subseteq X(u)$. For every $y \in X(u)$, $\{u, x, y\}$ is an interval of $G(X \cup \{x, y\})$. Moreover, for every $y \in (V \setminus X) \setminus X(u)$, $y \notin I$ and thus $I \cap (X \cup \{x, y\}) = \{u, x\}$ is an interval of $G(X \cup \{x, y\})$. It follows that for every $y \in (V \setminus X) \setminus \{x\}$, $\{x, y\} \notin E_X$. $\square$

Lastly, we recall the characterization of the critical graphs and some of their properties.

Theorem 2.8 (Schmerl and Trotter [5]). *Given $G = (V, E)$, an indecomposable graph with $|V| \geq 2$, G is critical if and only if G is isomorphic to G_{2n} or to $\overline{G_{2n}}$, where $n \geq 2$.*

Let $G = (V, E)$ be a graph with $|V| \geq 3$. The *indecomposability graph* of G ([3]) is the graph $\mathrm{Ind}[G]$ defined on V as follows: given $x \neq y \in V$, $\{x, y\}$ is an edge of $\mathrm{Ind}[G]$ if $G - \{x, y\}$ is indecomposable. An *indecomposability matching* of G is a matching $\mathcal{M}$ of $\mathrm{Ind}[G]$ such that for every nonempty subset $\mathcal{N}$ of $\mathcal{M}$, $G(\cup\mathcal{N})$ is indecomposable, where $\cup\mathcal{N}$ denotes the union of the elements of $\mathcal{N}$. Consider a subset X of V satisfying $|X| \geq 4$, $|V \setminus X| \geq 2$ and $G(X)$ is indecomposable. An *indecomposability matching* of G *according to* $G(X)$ is a matching $\mathcal{M}$ of G_X such that for every subset $\mathcal{N}$ of $\mathcal{M}$, $G(X \cup (\cup\mathcal{N}))$ is indecomposable.

We use the following properties of the critical graph G_{2n}.

Observation 2.9. *Consider an integer $n \geq 2$. The following assertions are easy to verify.*

(1) *G_{2n} is bipartite and*
$$B(G_{2n}) = \{\{2i; i \in \{0, \ldots, n-1\}\}, \{2i+1; i \in \{0, \ldots, n-1\}\}\}$$
is the corresponding bipartition.

(2) $\mathrm{Aut}(G_{2n}) = \{\mathrm{Id}_{\{0,\ldots,2n-1\}}, f_{2n}\}$, *where f_{2n} denotes the permutation of $\{0, \ldots, 2n-1\}$ which interchanges i and $(2n-1) - i$ for $i \in \{0, \ldots, 2n-1\}$.*

(3) *For any distinct elements i and j of $\{0, \ldots, 2n-1\}$, if $G_{2n} - \{i, j\}$ is not connected, then $G_{2n} - \{i, j\}$ admits isolated vertices.*

(4) *For every subset X of $\{0, \ldots, 2n-1\}$, with $|X| \geq 4$, if $G_{2n}(X)$ is indecomposable, then $G_{2n}(X)$ is critical and hence isomorphic to G_{2m}, where $m = |X|/2$. Furthermore, when $n \geq 3$, consider a subsequence $i_0 < \cdots < i_{p-1}$ of $\{0, \ldots, n-1\}$ with $p \geq 2$. The subgraph $G_{2n}(\{2i_0, 2i_0+1, \ldots, 2i_{p-1}, 2i_{p-1}+1\})$ is isomorphic to G_{2p}. It suffices to consider the function defined on $\{2i_0, 2i_0+1, \ldots, 2i_{p-1}, 2i_{p-1}+1\}$ by $2i_j \mapsto 2j$ and $2i_j + 1 \mapsto 2j + 1$ for $j \in \{0, \ldots, p-1\}$.*

(5) *For every* $j \in \{0, \dots, 2n-1\}$*,* $G_{2n} - j$ *admits a single non trivial interval* I_j *determined by:* $I_0 = \{2, \dots, 2n-1\}$*,* $I_{2n-1} = \{0, \dots, 2n-3\}$ *and* $I_j = \{j-1, j+1\}$ *for* $1 \leq j \leq 2n-2$*.*
(6) *When* $n \geq 3$*, we have* $\mathrm{Ind}[G_{2n}] = P_{2n}$ *so that* $\{\{2i, 2i+1\};\ 0 \leq i \leq n-1\}$ *is the unique matching of* $\mathrm{Ind}[G_{2n}]$*. It follows from the fourth assertion that* $\{\{2i, 2i+1\}; 0 \leq i \leq n-1\}$ *is the single indecomposability matching of* G_{2n}*.*

Let G be a critical graph. Then by Theorem 2.8 there exists an isomorphism f from G onto G_{2n} or $\overline{G_{2n}}$, where $n \geq 2$. Since $\mathrm{Aut}(G_{2n}) = \{\mathrm{Id}_{\{0,\dots,2n-1\}}, f_{2n}\}$ and since f_{2n} interchanges $\{2i; i \in \{0, \dots, n-1\}\}$ and $\{2i+1; i \in \{0, \dots, n-1\}\}$, the bipartition

$$\left\{f^{-1}\left(\{2i; i \in \{0, \dots, n-1\}\}\right), f^{-1}\left(\{2i+1; i \in \{0, \dots, n-1\}\}\right)\right\}$$

does not depend on the isomorphism f. It is denoted by $B(G)$.

Given an indecomposable graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. In this paper we characterize the graph G that is critical according to $G(X)$, by using the partitions p_X and q_X of $V \setminus X$ together with the criticality of the subgraphs of the graph G_X induced by the connected components of G_X.

3. Two examples

For our first example, as shown in Figure 2, we begin with the graph P_5 defined on $X = \{0, 1, 2, 3, 4\}$. Distinct elements $x_0, \dots, x_{2m-1}$ and $y_0, \dots, y_{2n-1}$, where $m, n \geq 1$, are added to X. We define the graph $G = (V, E)$, where

$$V = X \cup \{x_0, \dots, x_{2m-1}\} \cup \{y_0, \dots, y_{2n-1}\}$$

and

$$\begin{aligned} E = \{&\{0,1\}, \{1,2\}, \{2,3\}, \{3,4\}\} \\ &\cup \{\{1, x_i\}; i \in \{0, \dots, 2m-1\}\} \\ &\cup \{\{3, x_{2i+1}\}; i \in \{0, \dots, m-1\}\} \\ &\cup \{\{3, y_{2k}\}; k \in \{0, \dots, n-1\}\} \\ &\cup \{\{x_{2i}, x_{2j+1}\}; i \leq j \in \{0, \dots, m-1\}\} \\ &\cup \{\{y_{2k}, y_{2l+1}\}; k \leq l \in \{0, \dots, n-1\}\}. \end{aligned}$$

From our definition of G, we deduce the following assertions about the structure of G.

- $G(X) = P_5$;
- The function, defined on $\{0, \dots, 2m-1\}$, that associates x_i with each $i \in \{0, \dots, 2m-1\}$, realizes an isomorphism from G_{2m} onto $G(\{x_0, \dots, x_{2m-1}\})$;
- The function, defined on $\{0, \dots, 2n-1\}$, that associates y_k with each $k \in \{0, \dots, 2n-1\}$, realizes an isomorphism from G_{2n} onto $G(\{y_0, \dots, y_{2n-1}\})$;

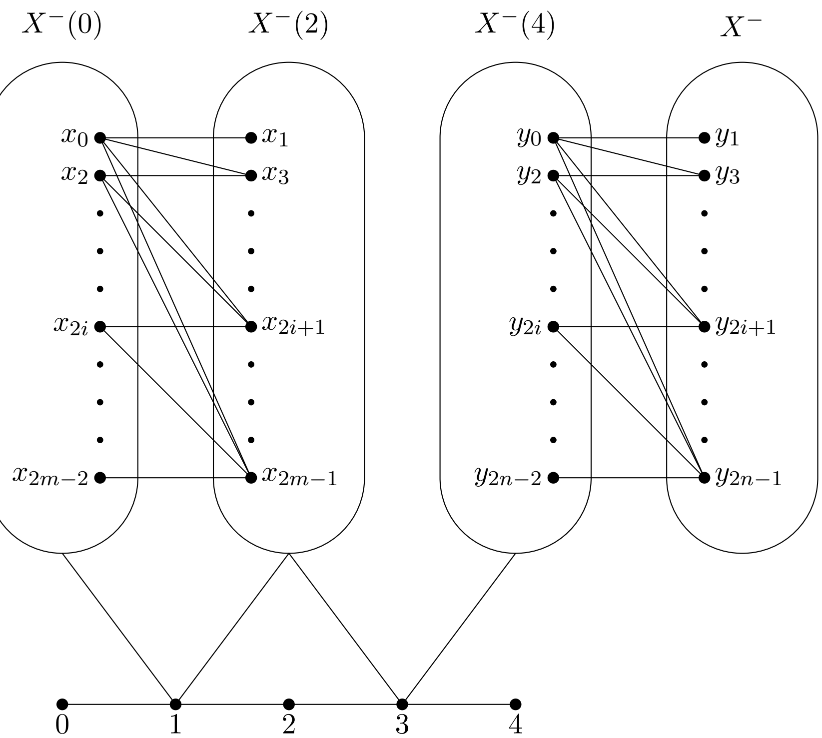

FIGURE 2. $p_X = q_X^- = \{X^-(0), X^-(2), X^-(4), X^-\}$ and $G_X = G - X$ has two connected components.

- $X^-(0) = \{x_{2i}; i \in \{0, \ldots, m-1\}\}$;
- $X^-(2) = \{x_{2i+1}; i \in \{0, \ldots, m-1\}\}$;
- $X^-(4) = \{y_{2k}; k \in \{0, \ldots, n-1\}\}$;
- $X^- = \{y_{2k+1}; k \in \{0, \ldots, n-1\}\}$;
- $p_X = q_X^- = \{X^-(0), X^-(2), X^-(4), X^-\}$;
- $G_X = G - X$.

Now, we want to show that the graph G is indecomposable and critical according to $G(X)$. We establish this in the following two claims.

Claim 3.1. *The graph G is indecomposable.*

Proof. Let I be an interval of G such that $|I| \geq 2$. First, if $I \cap X = \varnothing$, then I is included in $X^-(0)$, $X^-(2)$, $X^-(4)$ or X^-. Thus I would be a non-trivial interval either of $G(\{x_0, \ldots, x_{2m-1}\})$ or of $G(\{y_0, \ldots, y_{2n-1}\})$, which are indecomposable. Secondly, if $I \cap X = \{u\}$, then $u \in \{0, 2, 4\}$ and $I \setminus \{u\} \subseteq X^-(u)$. However, given $x \in I \setminus \{u\}$, there exists $y \in V \setminus (X \cup X^-(u))$ such that $\{x, y\} \in E$, which is impossible because $\{u, y\} \notin E$. It follows that $X \subseteq I$. If $x \in V \setminus I$, then, by Proposition 2.1, $I \cap (X \cup \{x\}) = X$ is an interval of $G(X \cup \{x\})$, that is, $x \in [X]$. Therefore, $X^-(0) \cup X^-(2) \cup X^-(4) \subseteq I$.

Given $k \in \{0, \ldots, n-1\}$, we have $\{y_{2k}, y_{2k+1}\} \in E$ but $\{0, y_{2k+1}\} \notin E$. As $0, y_{2k} \in I$, we obtain that $y_{2k+1} \in I$. Consequently, $I = V$. □

Claim 3.2. *The graph G is critical according to $G(X)$.*

Proof. It suffices to observe the following.

(1) $\{2, x_1\}$ is an interval of $G - x_0$ and $\{0, x_{2m-2}\}$ is an interval of $G - x_{2m-1}$;
(2) if $m \geq 2$, then for $1 \leq p \leq 2m-2$, $\{x_{p-1}, x_{p+1}\}$ is an interval of $G - x_p$;
(3) $V \setminus \{y_0, y_1\}$ is an interval of $G - y_0$ and $\{4, y_{2n-2}\}$ is an interval of $G - y_{2n-1}$;
(4) if $n \geq 2$, then for $1 \leq q \leq 2n-2$, $\{y_{q-1}, y_{q+1}\}$ is an interval of $G - y_q$.

□

For our second example, shown in Figure 3, we begin with the graph P_4 defined on $X = \{0, 1, 2, 3\}$. Distinct elements $x_0, \ldots, x_{2m-1}$ and $y_0, \ldots, y_{2n-1}$, where $m, n \geq 1$, are added to X. Thus, we define the graph $G = (V, E)$, where $V = X \cup \{x_0, \ldots, x_{2m-1}\} \cup \{y_0, \ldots, y_{2n-1}\}$ and

$$\begin{aligned} E = \{\{0,1\},\{1,2\},\{2,3\}\} &\cup \{\{1, x_{2i}\}; i \in \{0, \ldots, m-1\}\} \\ &\cup \{\{1, y_{2k}\}; k \in \{0, \ldots, n-1\}\} \\ &\cup \{\{x_{2i}, x_{2j+1}\}; i \leq j \in \{0, \ldots, m-1\}\} \\ &\cup \{\{y_{2k}, y_{2l+1}\}; k \leq l \in \{0, \ldots, n-1\}\}. \end{aligned}$$

The next assertions follow immediately:

- $G(X) = P_4$;
- The function, defined on $\{0, \ldots, 2m-1\}$, that associates x_i with each $i \in \{0, \ldots, 2m-1\}$, realizes an isomorphism from G_{2m} onto $G(\{x_0, \ldots, x_{2m-1}\})$;
- The function, defined on $\{0, \ldots, 2n-1\}$, that associates y_k with each $k \in \{0, \ldots, 2n-1\}$, realizes an isomorphism from G_{2n} onto $G(\{y_0, \ldots, y_{2n-1}\})$;
- $X^-(0) = \{x_{2i}; i \in \{0, \ldots, m-1\}\} \cup \{y_{2k}; k \in \{0, \ldots, n-1\}\}$;
- $X^- = \{x_{2i+1}; i \in \{0, \ldots, m-1\}\} \cup \{y_{2k+1};\ k \in \{0, \ldots, n-1\}\}$;
- $p_X = q_X^- = \{X^-(0), X^-\}$;
- $G_X = G - X$.

In Claims 3.3 and 3.4 below, we prove that G is indecomposable and critical according to $G(X)$.

Claim 3.3. *The graph G is indecomposable.*

Proof. Let I be an interval of G such that $|I| \geq 2$. First, if $I \cap X = \varnothing$, then I is included either in $X^-(0)$ or in X^-. If $I \subseteq X^-(0)$, then, since $G(\{x_0, \ldots, x_{2m-1}\})$ and $G(\{y_0, \ldots, y_{2n-1}\})$ are indecomposable, there exist $i \in \{0, \ldots, m-1\}$ and $k \in \{0, \ldots, n-1\}$ such that $I = \{x_{2i}, y_{2k}\}$. This is impossible because $\{x_{2i}, x_{2i+1}\} \in E$ and $\{y_{2k}, x_{2i+1}\} \notin E$. Similarly, if

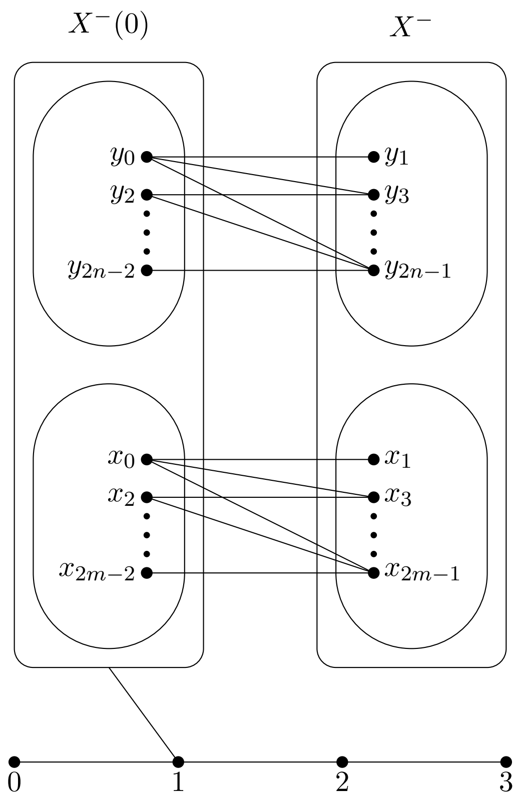

FIGURE 3. $p_X = q_X^- = \{X^-(0), X^-\}$ and $G_X = G - X$ has two connected components.

$I \subseteq X^-$, then, since $G(\{x_0, \ldots, x_{2m-1}\})$ and $G(\{y_0, \ldots, y_{2n-1}\})$ are indecomposable, there exist $i \in \{0, \ldots, m-1\}$ and $k \in \{0, \ldots, n-1\}$ such that $I = \{x_{2i+1}, y_{2k+1}\}$. This is impossible because $\{x_{2i+1}, x_{2i}\} \in E$ and $\{y_{2k+1}, x_{2i}\} \notin E$. Secondly, if $I \cap X = \{u\}$, then $u = 0$ and $I \setminus \{0\} \subseteq X^-(0)$. However, given $x \in I \setminus \{0\}$, there exists $y \in X^-$ such that $\{x, y\} \in E$, which is impossible because $\{0, y\} \notin E$. It follows that $X \subseteq I$. Therefore, $V \setminus I \subseteq [X]$ and thus $X \cup X^-(0) \subseteq I$. Given $i \in \{0, \ldots, m-1\}$, we have $\{x_{2i}, x_{2i+1}\} \in E$ and $\{0, x_{2i+1}\} \notin E$. As $0, x_{2i} \in I$, we obtain that $x_{2i+1} \in I$. Consequently, $\{x_0, \ldots, x_{2m-1}\} \subseteq I$. In the same manner, we verify that $\{y_0, \ldots, y_{2n-1}\} \subseteq I$. □

Claim 3.4. *The graph G is critical according to $G(X)$.*

Proof. It suffices to observe the following.

(1) $V \setminus \{x_0, x_1\}$ is an interval of $G - x_0$ and $\{0, x_{2m-2}\}$ is an interval of $G - x_{2m-1}$;
(2) if $m \geq 2$, then for $1 \leq p \leq 2m-2$, $\{x_{p-1}, x_{p+1}\}$ is an interval of $G - x_p$;

(3) $V \setminus \{y_0, y_1\}$ is an interval of $G - y_0$ and $\{0, y_{2n-2}\}$ is an interval of $G - y_{2n-1}$;
(4) if $n \geq 2$, then for $1 \leq q \leq 2n-2$, $\{y_{q-1}, y_{q+1}\}$ is an interval of $G - y_q$. □

4. The first properties

In the section, we consider an indecomposable graph $G = (V, E)$ and a proper subset X of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. Moreover, we assume that G is critical according to $G(X)$.

Lemma 4.1. *For every proper subset Y of V which includes X, if $G(Y)$ is indecomposable, then $|V \setminus Y|$ is even. In particular, $|V \setminus X|$ is even and* $\mathrm{Ext}(X) = \varnothing$.

Proof. By applying Corollary 2.4 several times, we obtain $G(Z)$ from $G(Y)$ such that $Y \subseteq Z \subset V$, $G(Z)$ is indecomposable and either $|V \setminus Z| = 1$ when $|V \setminus Y|$ is odd or $|V \setminus Z| = 2$ when $|V \setminus Y|$ is even. □

It follows from Remark 2.5 that G_X is multipartite by p_X. The next result is a simple consequence.

Proposition 4.2. *For every subset $X \subset Y \subseteq V$, if $G(Y)$ is indecomposable, then $G(Y)$ is critical according to $G(X)$.*

Proof. By the last lemma, $|V \setminus Y|$ is even. Consequently, for all $x \in Y \setminus X$, $|V \setminus (Y \setminus \{x\})|$ is odd and thus $G(Y) - x$ is decomposable. □

Lemma 4.3. *Given distinct elements a, b, c of $V \setminus X$, if $\{a, b\}, \{a, c\} \in E_X$, then $\{b, c\}$ is an interval of $G(X \cup \{a, b, c\})$ and hence there exists $M \in q_X$ such that $b, c \in M$.*

Proof. By the definition of $G_X = (V \setminus X, E_X)$, $G(X \cup \{a, b\})$ is indecomposable. By Lemma 4.1, $c \notin \mathrm{Ext}(Y)$, where $Y = X \cup \{a, b\}$. If $c \in [Y]$, that is, if Y is an interval of $G(Y \cup \{c\})$, then $Y \cap (X \cup \{a, c\}) = X \cup \{a\}$ is an interval of $G(X \cup \{a, c\})$, which contradicts $\{a, c\} \in E_X$. Therefore, there is $u \in Y$ such that $c \in Y(u)$, that means, $\{u, c\}$ is an interval of $G(Y \cup \{c\})$. Since $G(X \cup \{a, c\})$ is indecomposable, $\{u, c\} \cap (X \cup \{a, c\})$ is a trivial interval of $G(X \cup \{a, c\})$ and thus $u = b$. □

Lemma 4.4. *Given $M, N \in p_X$, consider $a \in M$ and $b \neq c \in N$ such that $\{a, b\} \in E_X$ and $\{a, c\} \notin E_X$. Then the following two assertions hold.*
(1) *If $N = [X]$, then $X \cup \{a, b\}$ is an interval of $G(X \cup \{a, b, c\})$.*
(2) *If $N = X(u)$, where $u \in X$, then $\{u, c\}$ is an interval of $G(X \cup \{a, b, c\})$.*

Proof. As observed in the preceding proof, where $Y = X \cup \{a, b\}$, $G(Y)$ is indecomposable and $c \notin \mathrm{Ext}(Y)$. As $\{a, b\} \in E_X$, it follows from Remark 2.5 that $M \neq N$. If $c \in Y(a)$, that is, if $\{a, c\}$ is an interval of $G(X \cup \{a, b, c\})$, then a and c would belong to the same element of p_X by

Lemma 2.6. Therefore, $c \notin Y(a)$. Furthermore, suppose that $c \in Y(b)$ or, equivalently, that $\{b,c\}$ is an interval of $G(X \cup \{a,b,c\})$. By Lemma 2.6 applied to $G(X \cup \{a,b,c\})$, $\{b,c\}$ would be an interval of $G_X(\{a,b,c\})$, which is impossible because $\{a,b\} \in E_X$ and $\{a,c\} \notin E_X$. It follows that either $c \in [Y]$ or $c \in Y(u)$, where $u \in X$. Clearly, we have $[Y] \subseteq [X]$ and $Y(u) \subseteq X(u)$ for every $u \in X$. Firstly, assume that $N = [X]$. Since p_X is a partition of $V \setminus X$, $c \notin X(u)$ and hence $c \notin Y(u)$ for every $u \in X$. It follows that $c \in [Y]$. Secondly, assume that $N = X(u)$, where $u \in X$. As $c \in X(u)$, $c \notin [X]$ and thus $c \notin [Y]$. Therefore, there is $v \in X$ such that $c \in Y(v)$. Consequently, $c \in X(v)$ and, as p_X is a partition of $V \setminus X$, $u = v$. □

Corollary 4.5. *The graph G_X has no isolated vertices.*

Proof. The family of the isolated vertices of G_X is denoted by W. Since $\mathrm{Ext}(X) = \varnothing$, it suffices to prove that $V \setminus ([X] \cap W)$ is an interval of G and $\{u\} \cup (W \cap X(u))$ is an interval of G for each $u \in X$. First, we claim that, given $x \in [X] \cap W$, $X \cup \{y\}$ is an interval of $G(X \cup \{x,y\})$ for every $y \in (V \setminus X) \setminus ([X] \cap W)$. Indeed, as x is isolated in G_X, $G(X \cup \{x,y\})$ is decomposable. If $y \notin [X]$, the claim follows from Lemma 2.3. Therefore, let $y \in [X] \setminus W$. As $y \notin W$, there is $z \in V \setminus X$ such that $\{y,z\} \in E_X$. Since $x \in W$, $\{x,z\} \notin E_X$ and the claim follows from Lemma 4.4. Secondly, we claim that, given $x \in W \cap X(u)$, $\{u,x\}$ is an interval of $G(X \cup \{x,y\})$ for every $y \in (V \setminus X) \setminus (W \cap X(u))$. Similar to our proof of the first claim, the second claim follows from Lemma 2.3 when $y \in (V \setminus X) \setminus X(u)$. Otherwise, Lemma 4.4 can be used by considering an element z such that $z \in V \setminus X$ and $\{y,z\} \in E_X$. □

Corollary 4.6. *The partitions p_X and q_X coincide.*

Proof. It is sufficient to show that $X^- = \varnothing$ or $X^+ = \varnothing$ and that for every $u \in X$, $X^-(u) = \varnothing$ or $X^+(u) = \varnothing$. By contradiction, suppose firstly that there are $a \in X^-$ and $b \in X^+$. By Corollary 4.5, there exist $c,d \in V \setminus X$ such that $\{a,c\},\{b,d\} \in E_X$. It follows from Lemma 4.3 that $\{a,d\},\{b,c\} \notin E_X$. In particular, $c \neq d$. By Lemma 4.4 applied to a,b,c, $X \cup \{a,c\}$ is an interval of $G(X \cup \{a,b,c\})$ and, since $b \in X^+$, $\{a,b\} \in E$. Similarly, by applying Lemma 4.4 to a,b,d, we have $X \cup \{b,d\}$ is an interval of $G(X \cup \{a,b,d\})$ and, since $a \in X^-$, $\{a,b\} \notin E$. Secondly, suppose that for $u \in X$, there exist $a \in X^-(u)$ and $b \in X^+(u)$. Again, consider $c \neq d \in V \setminus X$ such that $\{a,c\},\{b,d\} \in E_X$ and $\{a,d\},\{b,c\} \notin E_X$. We arrive at a contradiction similar to first claim. Indeed, by Lemma 4.4 applied to a,b,c, $\{u,b\}$ is an interval of $G(X \cup \{a,b,c\})$ and, since $a \in X^-(u)$, $\{a,b\} \notin E$. Again, by Lemma 4.4 applied to a,b,d, $\{u,a\}$ is an interval of $G(X \cup \{a,b,d\})$ and, since $b \in X^+(u)$, $\{a,b\} \in E$. □

Corollary 4.7. *For every $M \in q_X^-$, $G(M)$ is empty, and for every $M \in q_X^+$, $G(M)$ is complete.*

Proof. By interchanging the graph $G = (V,E)$ and its complement $\overline{G} = (V,\overline{E})$, we only need to consider the case for $M \in q_X^-$. It suffices to establish that each connected component M' of $G(M)$ is an interval of G. Clearly, M' is an interval of $G(M)$. Moreover, it follows from Corollary 4.6 that $M \in p_X$. Consequently, by Lemma 2.6, it is sufficient to show that M' is an interval of G_X. Since $G_X(M)$ is empty, M' is an interval of $G_X(M)$. Therefore, consider an element x of $(V \setminus X) \setminus M$. As $G(M')$ is connected, it suffices to verify that for $a, b \in M'$, with $\{a, b\} \in E$, $\{a, x\} \in E_X$ if and only if $\{b, x\} \in E_X$. Otherwise, there are $a, b \in M'$ such that $\{a, b\} \in E$, $\{a, x\} \in E_X$ and $\{b, x\} \notin E_X$. A contradiction follows from Lemma 4.4. That is, if $M = X^-$, then $X \cup \{a, x\}$ would be an interval of $G(X \cup \{a, b, x\})$ and thus $\{a, b\} \notin E$ and if $M = X^-(u)$, where $u \in X$, then $\{u, b\}$ would be an interval of $G(X \cup \{a, b, x\})$. This implies that $\{a, b\} \notin E$ because $a \in X^-(u)$. □

Discussion. *It follows that $G - X$ is entirely determined by p_X and by G_X. More precisely, let a and b be distinct elements of $V \setminus X$. Denote by M (resp. N) the element of p_X which contains a (resp. b). It follows from Corollary 4.6 that $M, N \in q_X$. Consequently, if $M = N$, then Corollary 4.7 is used. Indeed, either $M \in q_X^-$ and $\{a, b\} \notin E$ or $M \in q_X^+$ and $\{a, b\} \in E$. Now, suppose that $M \neq N$ and, for instance, that $M = X^-$ (resp. $M = X^+$). It follows that $\{a, b\} \in E$ if and only if $\{a, b\} \in E_X$ (resp. $\{a, b\} \notin E_X$). Finally, suppose that $M = X(u)$ and $N = X(v)$, where u and v are distinct elements of X such that $\{u, v\} \notin E$ (resp. $\{u, v\} \in E$). The same equivalences are obtained.*

Consequently, we examine the connected components of G_X in the next section.

5. The main results

Proposition 5.1. *Given an indecomposable graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. If G is critical according to $G(X)$, then for each connected component C of G_X, there exist distinct elements M_C and N_C of p_X satisfying $C \cap M_C \neq \varnothing$, $C \cap N_C \neq \varnothing$ and $C \subseteq M_C \cup N_C$.*

Proof. By Corollary 4.5, there exist $a \neq a' \in C$ such that $\{a, a'\} \in E_X$. By Remark 2.5, there are $M_C \neq N_C \in p_X$ such that $a \in M_C$ and $a' \in N_C$. For every $b \in C$, there exists a sequence $a = a_0, \ldots, a_n = b$ of elements of C such that $\{a_i, a_{i+1}\} \in E_X$ for $0 \leq i \leq n-1$. From Lemma 4.3 it follows that $a = a_0, a_2, \ldots \in M_C$ and $a_1, a_3, \ldots$ all belong to the same element of p_X. Therefore, it suffices to verify that $a_1 \in N_C$. It is obviously the case if $a_1 = a'$. Otherwise, Lemma 4.3 may be applied because $\{a, a'\}, \{a, a_1\} \in E_X$. □

Remark 5.2: Given an indecomposable graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. Assume that G is critical according to $G(X)$. It follows from Remark 2.5 and Proposition

5.1 that for each connected component C of G_X, $G_X(C)$ is bipartite by $\{C \cap M_C, C \cap N_C\}$.

Theorem 5.3. *Given a graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. The graph G is indecomposable and critical according to $G(X)$ if and only if the three assertions below are fulfilled.*

- H1: *The partitions p_X and q_X coincide.*
- H2: *For every $M \in q_X^-$, $G(M)$ is empty, and for $M \in q_X^+$, $G(M)$ is complete.*
- H3: *For each connected component C of G_X, $G(X \cup C)$ is indecomposable and critical according to $G(X)$.*

Proof. To begin, assume that G is indecomposable and critical according to $G(X)$. Assertions H1 and H2 are Corollaries 4.6 and 4.7 respectively. For Assertion H3, consider a connected component C of G_X. By Proposition 4.2, it is sufficient to establish that $G(X \cup C)$ is indecomposable. More precisely, we prove that if $G(X \cup C)$ admits a non trivial interval I, then I would be a non trivial interval of G as well. By Lemma 2.7 applied to $G(X \cup C)$, if $I \cap X \neq \varnothing$, then $G_X(C)$ admits isolated vertices. Consequently, C reduces to a singleton, which contradicts Corollary 4.5. It follows that $I \cap X = \varnothing$. By the last proposition and by Corollary 4.6, there exist $M_C \neq N_C \in q_X$ such that $C \subseteq M_C \cup N_C$. By Lemma 2.6 applied to $G(X \cup C)$, I is an interval of $G_X(C)$ and, for instance, $I \subseteq M_C$. As C is a connected component of G_X, I is an interval of G_X. Moreover, it follows from Corollary 4.7 that I is an interval of $G(M_C)$. Lastly, by Lemma 2.6 applied to G, I is an interval of G.

Conversely, we start with two simple observations. Firstly, a connected component C of G_X does not reduce to a singleton because $G(X \cup C)$ is critical according to $G(X)$. It follows that G_X does not have isolated vertices. Secondly, given $x \in V \setminus X$, denote by C the connected component of G_X which contains x. By Lemma 4.1, since $G(X \cup C)$ is critical according to $G(X)$, $G(X \cup \{x\})$ is decomposable, and thus, $\mathrm{Ext}(X) = \varnothing$. To continue, by contradiction suppose that G possesses a non trivial interval I. It follows from Lemma 2.7 that $I \cap X = \varnothing$. Let i be an element of I. Denote by C the connected component of G_X containing i. Since i is not an isolated vertex of G_X, there exists $x \in C$ such that $\{i, x\} \in E_X$. We obtain that $I \cap (X \cup \{i, x\})$ is a trivial interval of $G(X \cup \{i, x\})$ and necessarily $x \notin I$. It follows that $x \in C \setminus I$. By Lemma 2.6, I is an interval of G_X. Consequently, for $j \in I$, $\{j, x\} \in E_X$ and hence $j \in C$. It follows that $I \subseteq C \setminus \{x\}$ and I would be a non trivial interval of $G(X \cup C)$.

To complete the proof, we demonstrate that G is critical according to $G(X)$. Consider an element x of $V \setminus X$. Denote by C the connected component of G_X containing x. As $G(X \cup C)$ is critical according to $G(X)$, $G(X \cup C) - x$ admits a non trivial interval I. Firstly, if $X \subseteq I$, then $(C \setminus I) \setminus \{x\} \subseteq [X]$ and we verify that $V \setminus (C \setminus I)$ is an interval of $G - x$.

Given $y \in (C \setminus I) \setminus \{x\}$, we claim that for every $z \in V \setminus (C \setminus I)$, $X \cup \{z\}$ is an interval of $G(X \cup \{y, z\})$. If $z \in X \cup C$, then $I \cap (X \cup \{y, z\}) = X \cup \{z\}$ is an interval of $G(X \cup \{y, z\})$. If $z \in (V \setminus X) \setminus C$, then $\{y, z\} \notin E_X$, implying $G(X \cup \{y, z\})$ is decomposable. If $z \notin [X]$, then the claim follows from Lemma 2.3. If $z \in [X]$, then, by Hypothesis H1, either $y, z \in X^-$ or $y, z \in X^+$. It is then sufficient to apply Hypothesis H2. Secondly, if $I \cap X = \{u\}$, then $I \setminus \{u\} \subseteq X(u)$ and we prove that I is an interval of $G - x$. Given $y \in I \setminus \{u\}$, we claim that for every $z \in (V \setminus I) \setminus \{x\}$, $\{u, y\}$ is an interval of $G(X \cup \{y, z\})$. If $z \in X \cup C$, then $I \cap (X \cup \{y, z\}) = \{u, y\}$ is an interval of $G(X \cup \{y, z\})$. If $z \in (V \setminus X) \setminus C$, then $\{y, z\} \notin E_X$, that is, $G(X \cup \{y, z\})$ is decomposable. If $z \notin X(u)$, then the claim follows from Lemma 2.3. If $z \in X(u)$, then, by Hypothesis H1, either $y, z \in X^-(u)$ or $y, z \in X^+(u)$. Hypothesis H2 is then applied. Thirdly, if $I \cap X = \varnothing$, then we establish that I is an interval of $G - x$. By Lemma 2.6 applied to $G(X \cup C) - x$, there exists $N \in q_X$ such that $I \subseteq N$. Furthermore, I is an interval of $G_X(C) - x$. Since C is a connected component of G_X, I is an interval of $G_X - x$. By Hypothesis H2, I is an interval of $G(N) - x$. By Hypothesis H1, we have $N \in p_X$. Lastly, by applying Lemma 2.6 to $G - x$, I is an interval of $G - x$. □

Given a graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. We now characterize the cases where G is indecomposable and critical according to $G(X)$, assuming G_X is connected. If $|V \setminus X| = 2$, then the following is evident. The graph G is indecomposable and critical according to $G(X)$ if and only if $\mathrm{Ext}(X) = \varnothing$. Consequently, we assume that $|V \setminus X| \geq 3$.

Theorem 5.4. *Given a graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$, $|V \setminus X| \geq 3$, $G(X)$ is indecomposable and G_X is connected. The graph G is indecomposable and critical according to $G(X)$ if and only if the following four assertions are satisfied.*

- K1: $\mathrm{Ext}(X) = \varnothing$.
- K2: *The partitions p_X and q_X coincide.*
- K3: *For every $M \in q_X^-$, $G(M)$ is empty, and for $M \in q_X^+$, $G(M)$ is complete.*
- K4: *The graph G_X is critical and $B(G_X) = p_X$.*

Proof. To begin, assume that G is indecomposable and critical according to $G(X)$. Assertions K1, K2 and K3 are respectively Lemma 4.1 and Corollaries 4.6 and 4.7. Concerning the fourth assertion, K_4, we first prove that G_X is indecomposable. More precisely, we prove that if I is a non trivial interval of G_X, then I is a non trivial interval of G. By Proposition 5.1, p_X admits two elements denoted by M and N. By Remark 5.2, G_X is bipartite by $\{M, N\}$. Since G_X is connected, we have either $I \subseteq M$ or $I \subseteq N$. For instance, assume that $I \subseteq M$. By Corollary 4.7, I is an interval of $G(M)$ and, by Corollary 4.6, $M, N \in q_X$. By applying Lemma 2.6 to G, I is an interval of G. Now, we prove that G_X is critical. Given, for example, an element x of M, consider a non trivial interval I of $G-x$. By Lemma 2.7, if $I \cap X \neq \varnothing$, then $G_X - x$ admits isolated vertices and thus is decomposable. Consequently, assume that $I \cap X = \varnothing$. It follows from Lemma 2.6 applied to $G-x$ that either $I \subseteq M$ or $I \subseteq N$ and I is an interval of $G_X - x$.

Conversely, it follows from Hypotheses K2 and K4 that $B(G_X) = p_X = q_X$ has two elements denoted by M and N. By Hypothesis K1, $M \neq \mathrm{Ext}(X)$ and $N \neq \mathrm{Ext}(X)$. To begin, we verify that G is indecomposable. More precisely, we verify that if I is a non trivial interval of G, then I would be a non trivial interval of G_X also. It follows from Lemma 2.7 that $I \cap X = \varnothing$ and it suffices to apply Lemma 2.6. Therefore, we want to show that G is critical according to $G(X)$. Let x be an element of M. Following Observation 2.9.(5), we distinguish three cases. First, assume that there exists $y \in N$ such that y is an isolated vertex of $G_X - x$ and $N = [X]$. For each $x' \in M \setminus \{x\}$, $\{x', y\} \notin E_X$ and, by Lemma 2.3, $X \cup \{x'\}$ is an interval of $G(X \cup \{x', y\})$. For every $y' \in N \setminus \{y\}$, it follows from Hypothesis K3 that $X \cup \{y'\}$ is an interval of $G(X \cup \{y, y'\})$. It follows that $V \setminus \{x, y\}$ is an interval of $G-x$. Now, assume that there is $y \in N$ such that y is an isolated vertex of $G_X - x$ and $N = X(u)$, where $u \in X$. For each $x' \in M \setminus \{x\}$, $\{x', y\} \notin E_X$ and, by Lemma 2.3, $\{u, y\}$ is an interval of $G(X \cup \{x', y\})$. For every $y' \in N \setminus \{y\}$, it follows from Hypothesis K3 that $\{u, y\}$ is an interval of $G(X \cup \{y, y'\})$. It follows that $\{u, y\}$ is an interval of $G - x$. Lastly, assume that there are $y \neq z \in N$ such that $\{y, z\}$ is an interval of $G_X - x$. By Lemma 2.6 applied to $G - x$, $\{y, z\}$ is an interval of $G - x$. □

Given an indecomposable graph $G = (V, E)$, consider a proper subset X of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. Assume that G is critical according to $G(X)$. We use the following notation to describe the indecomposability matching of G according to $G(X)$. Consider a connected component C of G_X. By Theorem 5.3, $G(X \cup C)$ is indecomposable and critical according to $G(X)$. By Lemma 4.1 applied to $G(X \cup C)$, we have $|C|$ is even. Denote $|C|/2$ by $n(C)$. If $|C| = 2$, then we denote by g_C an isomorphism from P_2 onto $G_X(C)$. Assume that $|C| > 2$. By Theorem 5.4 applied to $G(X \cup C)$, there exists an isomorphism, denoted by g_C, from $G_{2n(C)}$ onto $G_X(C)$. Moreover, for any connected component C of G_X, $\mathcal{M}_C$ denotes the family $\{\{g_C(2i), g_C(2i+1)\};\ 0 \leq i \leq n(C) - 1\}$. Lastly, we denote by $\mathcal{M}_G$ the union of the families $\mathcal{M}_C$ over all the connected components C of G_X.

Proposition 5.5. *Given an indecomposable graph $G = (V, E)$, consider a proper subset X of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. If G is critical according to $G(X)$, then $\mathcal{M}_G$ is the unique indecomposability matching of G according to $G(X)$.*

Proof. Consider an indecomposability matching $\mathcal{M}$ of G according to $G(X)$. For any $x \neq y \in V \setminus X$ such that $\{x, y\} \in \mathcal{M}$, we have $G(X \cup \{x, y\})$ is indecomposable, that is, $\{x, y\} \in E_X$ and hence x and y belong to the same connected component of G_X. Given a connected component C of G_X, denote by $(\mathcal{M})_C$ the family of the elements of $\mathcal{M}$ contained in C. We proved that $\mathcal{M}$ is the union of the families $(\mathcal{M})_C$ over all the connected components C of G_X. Clearly, for each connected component C of G_X, $(\mathcal{M})_C$ is an indecomposability matching of $G(X \cup C)$ according to $G(X)$. To show that $\mathcal{M} = \mathcal{M}_G$, it suffices to verify that $(\mathcal{M})_C = \mathcal{M}_C$ for every connected component C of G_X. If $|C| = 2$, then $(\mathcal{M})_C = \{C\}$ and thus $(\mathcal{M})_C = \mathcal{M}_C$. Now, assume that $|C| > 2$. Let x and y be distinct elements of C such that $\{x, y\} \in (\mathcal{M})_C$. We have

$$G\left(X \cup \left(\cup((\mathcal{M})_C \setminus \{\{x, y\}\})\right)\right) = G(X \cup C) - \{x, y\}$$

is indecomposable. By Proposition 4.2, $G(X \cup C) - \{x, y\}$ is critical according to $G(X)$. Firstly, assume that $|C| = 4$ so that $G_X(C) - \{x, y\}$ has two vertices x' and y'. We have $G(X \cup C) - \{x, y\} = G(X \cup \{x', y'\})$ is indecomposable, that is, $\{x', y'\} \in E_X$. Since g_C is an isomorphism from G_4 onto $G_X(C)$, we obtain that $\{x, y\} = \{g_C(0), g_C(1)\}, \{g_C(1), g_C(2)\}$ or $\{g_C(2), g_C(3)\}$. Necessarily, $(\mathcal{M})_C = \{\{g_C(0), g_C(1)\}, \{g_C(2), g_C(3)\}\}$ because $(\mathcal{M})_C$ is a partition of C. Secondly, assume that $|C| > 4$. By Corollary 4.5 applied to $G(X \cup C) - \{x, y\}$, $G_X(C) - \{x, y\}$ does not have isolated vertices. By Observation 2.9.(3), $G_X(C) - \{x, y\}$ is connected because $G_X(C)$ is isomorphic to $G_{2n(C)}$. By Theorem 5.4 applied to $G(X \cup C) - \{x, y\}$, we have $G_X(C) - \{x, y\}$ is indecomposable. But, by Observation 2.9.(6), we have $\mathrm{Ind}[G_{2n(C)}] = P_{2n(C)}$. Consequently, there exists $i \in \{0, \ldots, 2n(C) - 2\}$ such that $\{x, y\} = \{g_C(i), g_C(i + 1)\}$. It follows that

$$(\mathcal{M})_C \subseteq \{\{g_C(i), g_C(i + 1)\};\ 0 \leq i \leq 2n(C) - 2\}.$$

Since $(\mathcal{M})_C$ is a partition of C, we obtain that $(\mathcal{M})_C = \mathcal{M}_C$.

To complete the proof, we establish that $\mathcal{M}_G$ constitutes an indecomposability matching of G according to $G(X)$. Given a nonempty subset $\mathcal{N}$ of $\mathcal{M}_G$, we have to show that $G(X \cup (\cup\mathcal{N}))$ is indecomposable. By applying Theorem 5.3 to $G(X \cup (\cup\mathcal{N}))$, it suffices to prove the following. For each connected component D of $G_X(\cup\mathcal{N})$, $G(X \cup D)$ is indecomposable and critical according to $G(X)$. By Proposition 4.2, it is sufficient to verify that $G(X \cup D)$ is indecomposable. As $G_X(D)$ is connected, there is a connected component C of G_X such that $D \subseteq C$. Therefore, $D \subseteq \cup\mathcal{N}_C$, where $\mathcal{N}_C$ denotes the family of the elements of $\mathcal{N}$ included in C. Clearly, if $|\mathcal{N}_C| = 1$, then $D = \cup\mathcal{N}_C$ and thus $G(X \cup D)$ is indecomposable. Consequently, assume that $|\mathcal{N}_C| \geq 2$. Since $G_X(C)$ is isomorphic to $G_{2n(C)}$, it follows from Observation 2.9.(4) that $G_X(\cup\mathcal{N}_C)$ is isomorphic to $G_{2p(C)}$, where $p(C) = |\mathcal{N}_C|/2$. In particular, $G_X(\cup\mathcal{N}_C)$ is connected and hence $D = \cup\mathcal{N}_C$. Furthermore, it follows from Theorem 5.4 applied to $G(X \cup (\cup\mathcal{N}_C))$ that $G(X \cup (\cup\mathcal{N}_C))$ is indecomposable. $\square$

The existence of such an indecomposability matching of G according to $G(X)$ is obtained in [1] without its uniqueness. An immediate consequence follows.

Corollary 5.6 (Deogun et al. [1]). *Given an indecomposable graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$ and $G(X)$ is indecomposable. If G is critical according to $G(X)$, then for each $x \in V \setminus X$, $G - x$ admits a unique non trivial interval I and either $|I| = 2$ or $|V \setminus I| = 2$.*

Proof. For every $x \in V \setminus X$, there exists $y \in V \setminus X$ such that $\{x, y\} \in \mathcal{M}_G$. Let $Y = V \setminus \{x, y\}$. Since $\mathcal{M}_G$ is an indecomposability matching of G according to $G(X)$, $G(X \cup (\cup(\mathcal{M}_G \setminus \{\{x, y\}\}))) = G(Y)$ is indecomposable. As G is critical according to $G(X)$, $G - x$ is decomposable, that is, $y \notin \mathrm{Ext}(Y)$. Therefore, $G - x$ admits a unique non trivial interval which is $V \setminus \{x, y\}$ if $y \in [Y]$ and $\{u, y\}$ if $y \in Y(u)$, where $u \in Y$. $\square$

Observation 5.7. *We can specify the non trivial interval I of $G - x$. Recall that we proved the following in the last part of the proof of Theorem 5.3. Denote by C the connected component of G_X which contains x. As $G(X \cup C)$ was assumed to be indecomposable and critical according to $G(X)$ by Assertion H3, $G(X \cup C) - x$ admits a non trivial interval J. Then, we established:*

- *if $X \subseteq J$, then $V \setminus (C \setminus J)$ is an interval of $G - x$;*
- *if $|X \cap J| \leq 1$, then J is an interval of $G - x$ as well.*

Assume that there is $y \in V \setminus X$ such that $C = \{x, y\}$. If $y \in [X]$, then $J = X$ and $I = V \setminus \{x, y\}$. If $y \in X(u)$, where $u \in X$, then $I = J = \{u, y\}$. Now, assume that $|C| \geq 4$. By Proposition 5.1 and Corollary 4.6, there exist $M_C \neq N_C \in q_X$ such that $C \subseteq M_C \cup N_C$. By Theorem 5.4 applied to $G(X \cup C)$, $G_X(C)$ is critical and $B(G_X(C)) = \{M_C \cap C, N_C \cap C\}$. Assume that $x \in M_C$. It follows from Observation 2.9 that if $G_X(C) - x$ admits an isolated vertex y, then $y \in N_C \cap C$. Otherwise, there are $y \neq z \in N_C \cap C$ such that $\{y, z\}$ is an interval of $G_X(C) - x$. In the last part of the proof of Theorem 5.4, we obtained the following interval J of $G(X \cup C) - x$.

- *If $G_X(C) - x$ admits an isolated vertex $y \in N_C \cap C$ and if $N_C = [X]$, then $(X \cup C) \setminus \{x, y\}$ is an interval of $G(X \cup C) - x$. Consequently, $I = V \setminus \{x, y\}$.*
- *If $G_X(C) - x$ admits an isolated vertex $y \in N_C \cap C$ and if $N_C = X(u)$, where $u \in X$, then $\{u, y\}$ is an interval of $G(X \cup C) - x$. Therefore, $I = \{u, y\}$.*
- *If there are $y \neq z \in N_C \cap C$ such that $\{y, z\}$ is an interval of $G_X(C) - x$, then $\{y, z\}$ is an interval of $G(X \cup C) - x$. It follows that $I = \{y, z\}$.*

To conclude, we compare the structure of the indecomposability graph of a critical graph and of a partially critical graph.

Corollary 5.8. *Given an indecomposable graph $G = (V, E)$, let X be a proper subset of V such that $|X| \geq 4$, $|V \setminus X| \geq 2$ and $G(X)$ is indecomposable. If G is critical according to $G(X)$, then* $\mathrm{Ind}[G] - X$ *and* G_X *share the same connected components. Furthermore, for each connected component* C *of* G_X*,* $\mathrm{Ind}[G](C)$ *is isomorphic to the path* $P_{|C|}$ *and* $\mathrm{Ind}[G](C) = \mathrm{Ind}[G_X(C)]$ *when* $|C| > 4$.

Proof. If $|V \setminus X| = 2$, then $V \setminus X$ is the unique edge of $\mathrm{Ind}[G] - X$ and of G_X. Consequently, assume that $|V \setminus X| > 2$. Given $x \in V \setminus X$, denote by C the connected component of G_X containing x. Consider an element x' of $V \setminus X$ such that $\{x, x'\}$ is an edge of $\mathrm{Ind}[G]$. The next three cases follow from Observation 5.7. Firstly, there exists $y \in C$ such that $V \setminus \{x, y\}$ is an interval of $G - x$. Necessarily, $x' = y$. Secondly, there are $y \neq z \in C$ such that $\{y, z\}$ is an interval of $G - x$. Consequently, $x' = y$ or z. Thirdly, there exist $y \in C$ and $u \in X$ such that $\{u, x\}$ is an interval of $G - x$. As $x' \notin X$, we have $x' = y$. In the three cases, we obtain that $x' \in C$. Therefore, each connected component of $\mathrm{Ind}[G] - X$ is contained in a connected component of G_X.

Consider a connected component C of G_X. Let x and x' be distinct elements of C. By Proposition 4.2, $\{x, x'\}$ is an edge of $\mathrm{Ind}[G]$ if and only if $G - \{x, x'\}$ is indecomposable and critical according to $G(X)$. Clearly, the connected components of $G_X - \{x, x'\}$ are the connected components of G_X distinct from C together with the connected components of $G_X(C) - \{x, x'\}$. By Theorem 5.3, for every connected component D of G_X such that $D \neq C$, $G(X \cup D)$ is indecomposable and critical according to $G(X)$. In particular, if $C = \{x, x'\}$, then $\{x, x'\}$ is an edge of $\mathrm{Ind}[G]$ by applying Theorem 5.3 to $G - \{x, x'\}$. Therefore, assume that $|C| > 2$. By Theorem 5.3 again, we deduce that $\{x, x'\}$ is an edge of $\mathrm{Ind}[G]$ if and only if for every connected component C' of $G_X(C) - \{x, x'\}$, $G(X \cup C')$ is indecomposable and critical according to $G(X)$ or, equivalently by Proposition 4.2, $G(X \cup C')$ is indecomposable. Consequently, by assuming that $G_X(C) - \{x, x'\}$ is connected, we have: $\{x, x'\}$ is an edge of $\mathrm{Ind}[G]$ if and only if $G(X \cup C) - \{x, x'\}$ is indecomposable. Firstly, assume that $|C| = 4$ so that $G_X(C) - \{x, x'\}$ possesses two vertices y and y'. We obtain that $\{x, x'\}$ is an edge of $\mathrm{Ind}[G]$ if and only if $\{y, y'\} \in E_X$. Since g_C is an isomorphism from G_4 onto $G_X(C)$, $\{y, y'\} \in E_X$ if and only if $\{x, x'\} = \{g_C(0), g_C(1)\}, \{g_C(1), g_C(2)\}$ or $\{g_C(2), g_C(3)\}$. Secondly, assume that $|C| > 4$. To begin, assume that $\{x, x'\}$ is an edge of $\mathrm{Ind}[G]$. By Corollary 4.5 applied to $G - \{x, x'\}$, $G_X - \{x, x'\}$ and thus $G_X(C) - \{x, x'\}$ do not have isolated vertices. Since $G_X(C)$ is isomorphic to $G_{2n(C)}$, it follows from Observation 2.9.(3) that $G_X(C) - \{x, x'\}$ is connected. As previously seen, we deduce from Theorem 5.3 applied to $G - \{x, x'\}$ that $G(X \cup C) - \{x, x'\}$ is indecomposable and critical according to $G(X)$. By Theorem 5.4 applied to $G(X \cup C) - \{x, x'\}$, $G_X(C) - \{x, x'\}$ is critical. In particular, $G_X(C) - \{x, x'\}$ is indecomposable, that is, $\{x, x'\}$ is an edge of $\mathrm{Ind}[G_X(C)]$. Conversely, assume that $\{x, x'\}$ is an edge of $\mathrm{Ind}[G_X(C)]$.

By Observation 2.9.(4), $G_X(C) - \{x, x'\}$ is critical. In particular, $G_X(C) - \{x, x'\}$ is connected. It results from Theorem 5.4 applied to $G(X \cup C) - \{x, x'\}$ that $G(X \cup C) - \{x, x'\}$ is indecomposable and critical according to $G(X)$. Finally, it follows from Theorem 5.3 applied to $G - \{x, x'\}$ that $G - \{x, x'\}$ is indecomposable. □

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Department of Math/CS, Nebraska Wesleyan University,
5000 St. Paul Ave., Lincoln NE, 68504, U.S.A.
E-mail address: abreiner@nebrwesleyan.edu

Department of Computer Science and Engineering,
University of Nebraska–Lincoln,
Lincoln NE, 68588-0115, U.S.A.
E-mail address: deogun@cse.unl.edu

Institut de Mathématiques de Luminy CNRS-UPR 9016,
163 avenue de Luminy - Case 907, 13288 Marseille Cedex 09, France
E-mail address: ille@iml.univ-mrs.fr