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CONNECTED DOMINATION DOT-CRITICAL GRAPHS

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Abstract. A dominating set in a graph $G = (V(G), E(G))$ is a set D of vertices such that every vertex in $V(G) \setminus D$ has a neighbor in D. A connected dominating set of a graph G is a dominating set whose induce subgraph is connected. The connected domination number $\gamma_c(G)$ is the minimum number of vertices of a connected dominating set of G. A graph G is connected domination dot-critical (cdd-critical) if contracting any two adjacent vertices decreases $\gamma_c(G)$; and G is totally connected domination dot-critical (tcdd-critical) if contracting any two vertices decreases $\gamma_c(G)$. We provide characterizations of tcdd-critical graphs for the classes of block graphs, split graphs and unicyclic graphs and a characterization of cdd-critical cacti.

1. Introduction

We consider finite, undirected, simple graphs. Given a graph G with vertex set $V(G)$ and edge set $E(G)$, a set $D \subseteq V(G)$ is a *dominating set* if every vertex of $V(G) \setminus D$ has a neighbor in D; and D is a *connected dominating set (cds)* if it is a dominating set and the subgraph induced by D is connected. The *connected domination number* $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G.

The *contraction* of two vertices u, v in a graph G is the graph G_{uv} obtained from G by removing u and v and adding a new vertex, denoted $\overline{uv}$, adjacent to every vertex of $G \setminus \{u, v\}$ that is adjacent to u or v.

A graph G is said to be *connected domination dot-critical*, abbreviated *cdd-critical*, if contracting any two adjacent vertices decreases $\gamma_c(G)$; and G is *totally connected domination dot-critical*, abbreviated *tcdd-critical*, if contracting any two vertices decreases $\gamma_c(G)$. Domination dot-critical graphs were introduced by Burton and Sumner [1, 2].

In this paper, we investigate (totally) connected domination dot-critical graphs in several classes of graphs: cacti, unicyclic graphs, split graphs. Before proceeding to the main results, we need to introduce some additional definitions and notations. The number of vertices $|V(G)|$ of a graph G is

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called the *order* of G. The neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, and the degree of v is $d_G(v) = |N(v)|$. A vertex of degree one is called a *leaf*, and the neighbor of any leaf is called a *support vertex*. A vertex is a *cut vertex* if its removal from G increases the number of components. A connected subgraph B of G is a *block* of G if the graph B has no cut vertex and is maximal with this property among all subgraphs of G. An *end block* is a block that contains at most one cut vertex of G. It is well-known and easy to see that the blocks vs. cut vertices incidence graph of any graph is acyclic, and consequently that every graph has an end block. A graph G is a *block graph* if every block of G is complete. A graph G is a *cactus* if each edge of G is contained in at most one cycle, equivalently, if every block of G is a cycle or an edge. A cactus that has only one cycle is called a *unicyclic graph*, and a connected cactus with no cycle is a *tree*.

To simplify notation, when we are dealing with a graph G and a contraction G_{uv} of G, we will frequently identify a set $S \subseteq V(G)$ with a set $S' \subseteq V(G_{uv})$ whenever the following happens: either S does not contain any of u, v and $S' = S$, or S contains at least one of u, v and S' contains $\overline{uv}$. In that case we may also use the same name for the two sets.

2. Preliminary results

We begin by some observations.

Observation 2.1. *Every connected dominating set contains every cut vertex.*

Observation 2.2. *Let G be a connected graph and u, v any two vertices of G. Then:*

(a) *If $uv \in E$, then $\gamma_c(G) - 1 \leq \gamma_c(G_{uv}) \leq \gamma_c(G)$.*
(b) *If $uv \notin E$, then $\gamma_c(G) - 3 \leq \gamma_c(G_{uv}) \leq \gamma_c(G)$.*

Proof. Clearly every $\gamma_c(G)$-set is a cds of G_{uv} and so $\gamma_c(G_{uv}) \leq \gamma_c(G)$ for items (a) and (b). Now let D be a $\gamma_c(G_{uv})$-set.

Assume that $uv \in E$. If $\overline{uv} \notin D$, then either D, $D \cup \{v\}$ or $D \cup \{u\}$ is a cds of G. If $\overline{uv} \in D$, then $\{u, v\} \cup D \setminus \{\overline{uv}\}$ is a cds of G. In any case we obtain $\gamma_c(G) \leq \gamma_c(G_{uv}) + 1$.

Assume now that $uv \notin E(G)$. If $\overline{uv} \notin D$, then at least one of u and v, say v, is adjacent to D. Since G is a connected graph, then u has either a neighbor in D or is adjacent to some vertex $w \in V \setminus D$ which is adjacent to D. Then D or $D \cup \{w\}$ is a cds of G, respectively. It follows that $\gamma_c(G) \leq \gamma_c(G_{uv}) + 1$.

Now let $\overline{uv} \in D$. If the subgraph induced by $\{u, v\} \cup D \setminus \{\overline{uv}\}$ is connected, then $\{u, v\} \cup D \setminus \{\overline{uv}\}$ is a cds of G and hence $\gamma_c(G) \leq \gamma_c(G_{uv}) + 1$. Now suppose that $D'' = \{u, v\} \cup D \setminus \{\overline{uv}\}$ has two components H_u and H_v. Since D is connected, we have $u \in H_u$ and $v \in H_v$. Since G is connected, there exists in $V \setminus D$ either one vertex z adjacent to H_u and H_v or two adjacent vertices x and y such that x is adjacent to H_u and y is adjacent to H_v, but then $D'' \cup \{z\}$ or $D'' \cup \{x, y\}$ is a cds of G, which implies that $\gamma_c(G) \leq \gamma_c(G_{uv}) + 3$. □

Note that the left inequality in item (b) of Observation 2.2 is attained for a path P_8 by contracting the support vertices.

Observation 2.3. *Let G be a graph and D be any $\gamma_c(G)$-set of G. Let C be a cycle that is also a block of G, and let $A_C = \{x \in C \mid d_G(x) = 2\}$. Then D contains at least $|V(C)| - 2$ vertices of C. Moreover:*

- *D contains all vertices of C if and only if every vertex of C is a cut vertex.*
- *D contains exactly $|V(C)| - 1$ vertices of C if and only if A_C is a non-empty independent set. In that case, for every vertex $a \in A_C$ there is a $\gamma_c(G)$-set D of G such that $D \cap V(C) = V(C) \setminus \{a\}$.*
- *D contains exactly $|V(C)|-2$ vertices of C if and only if A_C contains two adjacent vertices. In that case, for any two adjacent vertices $a, b \in A_C$ there is a $\gamma_c(G)$-set D of G such that $D \cap V(C) = V(C) \setminus \{a, b\}$.*
- *If a vertex x of C is not contained in any $\gamma_c(G)$-set, then either $A_C = \{x\}$, or $A_C = \{x, y\}$ where x, y are adjacent, or A_C contains three consecutive vertices with x in the middle and the remaining vertices of A_C form an independent set.*

Proof. Note that every vertex of C is either a cut vertex or a member of A_C. If D contains at most $|V(C)| - 3$ vertices of C, then either D is not connected or some vertex of C has no neighbor in D. This proves that D contains at least $|V(C)| - 2$ vertices of C. Moreover, by Observation 2.1, the first three items follow easily. Finally the last item follows easily from the first three. □

Proposition 2.4. *Let H be a block of a cdd-critical graph G.*

1. *If H is a cycle, let $A_H = \{x \in H : d_G(x) = 2\}$. Then:*
   - *Every support vertex in H has degree three,*
   - *$|A_H| \neq 1$.*
   - *If H contains a support vertex, then A_H is an independent set. Moreover, if $|A_H| \geq 2$, then every support vertex of H is adjacent to a vertex of A_H.*
   - *If H does not contain any support vertex, then the subgraph induced by A_H has either zero or at least two edges.*
2. *If H is a complete graph, then:*
   - *If H is an end block, then $H = P_2$,*
   - *No two end blocks have a common vertex,*
   - *If H is not an end block, then every vertex of H is a cut vertex.*

Proof. (1): First let H be a cycle.

Let x be a support vertex of H and x' be a leaf adjacent to x. If x has degree at least four, then $\overline{xx'}$ is a cut vertex in $G_{xx'}$, and so every $\gamma_c(G_{xx'})$-set is a connected dominating set of G, which implies $\gamma_c(G) \leq \gamma_c(G_{xx'})$, a contradiction. So x has degree three.

Suppose that $|A_H| = 1$. Then the remaining vertices of H are cut vertices. So by contracting the unique vertex of A_H with one of its two neighbors, it is easy to see that every minimum cds of the resulting graph is a cds for G, a contradiction.

Suppose now that H contains a support vertex x (with leaf neighbor x'). If A_H is not an independent set, then, by Observation 2.3, every $\gamma_c(G_{xx'})$-set D contains exactly $|V(H)| - 2$ vertices from H, with $\overline{xx'} \in D$. But then $\{x\} \cup D \setminus \{\overline{xx'}\}$ is a cds of G. So A_H is an independent set. Moreover, suppose that $|A_H| \geq 2$ and x is not adjacent to any vertex of A_H. Then the two neighbors of x in H are cut vertices, and so $A_H \cup \{\overline{xx'}\}$ is independent. Consequently every $\gamma_c(G_{xx'})$-set D contains exactly $|V(H)|-1$ vertices from H, with $\overline{xx'} \in D$, but then $\{x\} \cup D \setminus \{\overline{xx'}\}$ is a cds of G, a contradiction. Thus every support vertex in C is adjacent to A_H.

Finally, suppose that H does not contain any support vertex and that the subgraph induced by A_H contains exactly one edge, say ab. Then every $\gamma_c(G_{ab})$-set D contains all vertices of the resulting cycle except one, say $\overline{ab}$, but then D is a cds of G, a contradiction.

(2): Now let H be a complete graph. If H is an end block of order at least three, then by contracting any two vertices of H, we see that every minimum cds of the new graph is a cds of G, a contradiction. Next, if H_1 and H_2 are two end blocks with a common vertex, then contracting the edge of H_1 will not decrease $\gamma_c(G)$. Finally, if y is a vertex of a non-end block H and y is not a cut vertex, then contracting y with any vertex of $V(H) \setminus \{y\}$ will not decrease $\gamma_c(G)$. □

3. (T)CDD-CRITICAL GRAPHS

Since $\gamma_c(G) \geq 1$ for every connected graph, it is obvious that there is no cdd-critical graph with $\gamma_c(G) = 1$. From now on all graphs considered satisfy $\gamma_c(G) \geq 2$. We first give a sufficient condition for a 2-connected graph to be tcdd-critical.

Proposition 3.1. *Let G be a 2-connected graph such that for every vertex v, $\gamma_c(G \setminus v) < \gamma_c(G)$. Then G is a tcdd-critical graph.*

Proof. Let u and v be any two vertices of G and let S be a $\gamma_c(G \setminus v)$-set. If $u \notin S$, then u has a neighbor in S and S is cds of G_{uv}. If $u \in S$, then u and v are not adjacent for otherwise S would be a cds of G of size less than $\gamma_c(G)$, and so $\{\overline{uv}\} \cup S \setminus \{u\}$ is a cds of G_{uv}. In each case $\gamma_c(G_{uv}) < \gamma_c(G)$ and G is tcdd-critical. □

Note that the converse of Proposition 3.1 is not true in general. To see this, consider the graph that consists in two triangles abc, def plus two vertices g, h and edges ag, gd, ch, hf and be.

Theorem 3.2. *Let G be a connected block graph. The following statements are equivalent:*

(a) *G is a tcdd-critical graph.*
(b) *G is a cdd-critical graph.*
(c) *Every block H of G satisfies:*
- *If H is an end block, then $H = P_2$.*
- *If H is not an end block, then every vertex of H is a cut vertex.*
- *Every support vertex belongs to exactly two blocks.*

Proof. (a) $\Rightarrow$ (b): This implication is obvious.

(b) $\Rightarrow$ (c): By Proposition 2.4(2), we have the first two items of (c). Now let u be a support vertex that belongs to three blocks, and let v be a leaf neighbor of u. Then u is in every $\gamma_c(G_{uv})$-set D and so D is a cds of G, a contradiction.

(c) $\Rightarrow$ (a): We first note that the connected domination number of a block graph G whose blocks satisfy condition (c) is $\gamma_c(G) = n - |L(G)|$, where $L(G)$ is the set of leaves of G (note that, by (c), $|L(G)|$ is also the number of end blocks of G). Let u, v be any two vertices of G. Assume that u and v are adjacent. If each of u, v has degree at least two, then $L(G) = L(G_{uv})$, and so $\gamma_c(G_{uv}) \leq (n-1) - |L(G_{uv})| < n - |L(G)| = \gamma_c(G)$. Now assume that v is the leaf adjacent to u. Since u belongs to two blocks, $\overline{uv}$ is contained in a unique block of G_{uv}. If D is a $\gamma_c(G)$-set, then $u \in D$ and $D \setminus \{u\}$ is a cds of G_{uv}.

Assume now that u and v are not adjacent. If u, v are not leaves, then $V(G_{uv}) \setminus L(G_{uv})$ is a connected dominating set of G_{uv} of size $(n-1) - |L(G_{uv})| < \gamma_c(G)$. Assume now that at least one of u, v, say v, is a leaf and let w be the support vertex of v. Then for every $\gamma_c(G)$-set D, $D \setminus \{w\}$ is a cds of G_{uv} of size less than $|D|$. In all cases, $\gamma_c(G_{uv}) < \gamma_c(G)$ and so G is tcdd-critical. □

Corollary 3.3. *Let T be a tree of order $n \geq 4$. The following statements are equivalent:*

(a) *T is tcdd-critical tree.*
(b) *T is cdd-critical tree.*
(c) *Every support vertex of T has degree two.*

Recall that a *split graph* is a graph whose vertex-set can be partitioned into a clique and an independent set. The *corona graph* $G \circ K_1$ of a graph G is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and an edge vv' are added.

Theorem 3.4. *Let G be a connected split graph. Then the following statements are equivalent:*

(a) *G is tcdd-critical graph.*
(b) *G is cdd-critical graph.*
(c) *G is the corona of a complete graph with at least two vertices.*

Proof. Implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are obvious.

(b) $\Rightarrow$ (c): Let G be a split graph, partitioned into a clique Q and an independent set I, and let D be any $\gamma_c(G)$-set. Clearly $D \subseteq Q \neq \varnothing$ and so $|Q| \geq 2$. Since the subgraph induced by D is a clique, each vertex v of D has at least one private neighbor in $V \setminus D$, and hence in I. Moreover, such a private neighbor is unique, for otherwise contracting the edge with v does not decrease $\gamma_c(G)$, a contradiction. Also we have $|Q \setminus D| \leq 1$, for otherwise contracting any two vertices in $Q \setminus D$ does not decrease $\gamma_c(G)$. If there is a vertex y of I that is adjacent to at least two vertices of D, then contacting any edge between y and $N(y) \cap D$ does not decrease $\gamma_c(G)$. Thus I is the set of private neighbors of D. If $Q \setminus D = \varnothing$, then each vertex of I is a pendent vertex and so G is the corona of Q. Now suppose that $Q \setminus D = \{z\}$. If z is adjacent to two vertices x' and y' of I, then $\{z\} \cup D \setminus \{x, y\}$ is a cds of G of size less than D, where x and y are the neighbors of x' and y' in D, respectively. Now contracting any edge between z and a vertex of D does not decrease $\gamma_c(G)$, a contradiction. Thus $Q \setminus D = \varnothing$. □

Theorem 3.5. *Let G be a connected cactus. Then G is cdd-critical graph if and only if the following hold:*

(a) *Every support vertex that belongs to a cycle of G has degree three, and every support vertex that does not belong to any cycle has degree two.*
(b) *For every cycle C of G, the set $A_C = \{x \in C : d_G(x) = 2\}$ has size different from 1 and:*
   (i) *If C contains a support vertex, then A_C is an independent set and if moreover $|A_C| \geq 2$, then every support vertex of C is adjacent to a vertex of A_C.*
   (ii) *If C does not contain any support vertex, then the subgraph induced by A_C has either zero or at least two edges.*

Proof. Assume that G is cdd-critical. By Proposition 2.4, we have all items except the second part of item (a). Let u be a support vertex of degree at least three and v be a leaf adjacent to u. Then $\overline{uv}$ is a cut vertex of G_{uv}, and so every $\gamma_c(G_{uv})$-set is a cds of G. It follows that $\gamma_c(G) \leq \gamma_c(G_{uv})$, a contradiction.

Conversely assume that G satisfies items (a) and (b). Let x and y be any two adjacent vertices of G, and let D be any $\gamma_c(G)$-set. We distinguish between two cases.

Case 1: No cycle contains both x and y:
Clearly one of x and y is a cut vertex. If x and y are both cut vertices, then by Observation 2.1 they are in D and so $\{\overline{xy}\} \cup D \setminus \{x, y\}$ is a cds of G_{xy}. Thus $\gamma_c(G_{xy}) < \gamma_c(G)$. Now assume that x is not a cut

vertex, i.e., x is a leaf. If y is not in any cycle, then it is a support vertex of degree two. Thus $y \in D$ and so $D \setminus \{y\}$ is cds of G_{xy}. Hence $\gamma_c(G_{xy}) < \gamma_c(G)$. If y is in some cycle C, then by our hypothesis it has degree three. By item (b), A_C is independent. Thus D contains all vertices of C (if $A_C = \varnothing$) or all except one vertex z, and we may assume without loss of generality that z is a vertex of A_C adjacent to y (such a vertex exists by item (b)). Then $D \setminus \{y\}$ is a cds of G_{xy} and so $\gamma_c(G_{xy}) < \gamma_c(G)$.

CASE 2: x and y lie in a common cycle C:
If both are cut vertices, then $\{\overline{xy}\} \cup D \setminus \{x, y\}$ is a cds of G_{xy}, and so $\gamma_c(G_{xy}) < \gamma_c(G)$. If both are not cut vertices, then $x, y \in A_C$. Now A_C is not independent, so C contains no support vertex. By item (b.ii) there is another pair x', y' of adjacent vertices of A_C. Then D contains all vertices of C except two, and, by Observation 2.3, we may assume without loss of generality that these two are x' and y'. If edges xy and $x'y'$ are not adjacent, then $\{\overline{xy}\} \cup D \setminus \{x, y\}$ is a cds of G_{xy}. Else, we may assume that $x = x'$, and then $D \setminus \{y\}$ is a cds of G_{xy}. In both cases we have $\gamma_c(G_{xy}) < \gamma_c(G)$. Finally assume that x is not a cut vertex and y is a cut vertex. If C contains a support vertex t, then since x has degree two, by item (b), A_C is independent and contains another vertex of degree two, say a. Thus D contains all vertices of C except one, and we may assume without loss of generality that this exceptional vertex is a; and so $\{\overline{xy}\} \cup D \setminus \{x, y\}$ is a cds of G_{xy}. Suppose now that C contains no support vertex. If A_C is independent, then by item (b.ii), it contains another vertex $a \neq x$ and so D contains all vertices of C except a. Thus $\{\overline{xy}\} \cup D \setminus \{x, y\}$ is a cds of G_{xy}. If A_C is not independent, then by item (ii), it induces at least two edges. We may assume that $x \in D$ since D contains $V(C)$ except two vertices (which are any two adjacent vertices of A_C). Thus $\{\overline{xy}\} \cup D \setminus \{x, y\}$ is a cds of G_{xy}. In all cases above, we have $\gamma_c(G_{xy}) < \gamma_c(G)$, and so G is cdd-critical graph.

□

Theorem 3.6. *Let G be a connected unicyclic graph with cycle C, and let $A_C = \{x \in C : d_G(x) = 2\}$. Then G is tcdd-critical graph if and only if the following hold:*

(a) *Every support vertex in C has degree three and every support vertex not in C has degree two.*
(b) *If C contains a support vertex, then:*
   - *A_C is an independent set of size different from 1 and 2.*
   - *If $A_C \neq \varnothing$, then every support vertex of C is adjacent to a vertex of A_C.*

(c) *If C does not contain any support vertex, then the subgraph induced by A_C has either zero or at least two edges. Moreover if A_C is independent, then $|A_C| \geq 3$.*

Proof. Let G be a tcdd-critical unicyclic graph. Thus G is cdd-critical and so we have items (a) and (b) of Theorem 3.5. To get Theorem 3.6 we prove the following.

Assume that C contains a support vertex and let $A_C = \{u, v\}$. After contracting u and v, we see that all vertices of $\{\overline{uv}\} \cup V(C) \setminus \{u, v\}$ are cut vertices in G_{uv}. If D is any $\gamma_c(G_{uv})$-set, then $\{u\} \cup D \setminus \{\overline{uv}\}$ is a cds of G, a contradiction.

To complete the proof of (c) we use the same argument as above to see that if C does not contain any support vertex, then A_C is an independent set of size at least two. Moreover, if $|A_C| = 2$, say $A_C = \{u, v\}$, then every vertex of $\{\overline{uv}\} \cup V(C) \setminus \{u, v\}$ is a cut vertex of G_{uv}, so if S is any $\gamma_c(G_{uv})$-set then $\{u\} \cup S \setminus \{\overline{uv}\}$ is a cds of G, a contradiction. Thus $|A_C| \geq 3$.

Conversely, assume that G satisfies items (a)–(c). Clearly by Theorem 3.5 the contraction of any edge decreases $\gamma_c(G)$. Thus let u and v be two non-adjacent vertices of G. We distinguish between three cases.

Case 1: Both u and v are contained in some $\gamma_c(G)$-set D:

Then $\{\overline{uv}\} \cup D \setminus \{u, v\}$ is a cds of G_{uv} and so $\gamma_c(G_{uv}) < \gamma_c(G)$.

Case 2: No $\gamma_c(G)$-set contains any of u and v:

Then either both u and v are leaves or, by Observation 2.3 and items (b) and (c), u is a leaf and v is the middle of three consecutive vertices of degree two in C. Suppose that u and v are leaves and let u' be the support vertex of u. Note that if $u' \in C$, then by (b) every $\gamma_c(G)$-set contains either all vertices of C (when $A_C = \varnothing$) or $|V(C)| - 1$ vertices of C (when $A_C \neq \varnothing$). In the latter case, u' is adjacent to a vertex of A_C that we may assume does not belong to D. In either case, $D \setminus \{u'\}$ is a cds of G_{uv}. Now suppose that u is a leaf and v is the middle of three consecutive vertices of degree two in C. Since A_C is not independent, the support vertex u' adjacent to u does not belong to C. Thus for every $\gamma_c(G)$-set D, the set $D \setminus \{u'\}$ is a cds of G_{uv}. It follows that $\gamma_c(G_{uv}) < \gamma_c(G)$.

Case 3: u belongs to some $\gamma_c(G)$-set D, and no $\gamma_c(G)$-set contains v:

Thus v is either a leaf or the middle of three consecutive vertices of degree two in C, say a, v, b, and C contains no two adjacent vertices of $A_C \setminus \{a, v, b\}$. Suppose that v is a leaf and let v' be its support vertex. If $v' \notin C$, then $D \setminus \{v'\}$ is a cds of G_{uv}. If $v' \in C$, then, by (b), D contains either $|V(C)|$ or $|V(C)| - 1$ vertices of C depending on whether A_C is empty or not, respectively. Then $D \setminus \{v'\}$ is a cds of G_{uv}. Finally suppose that v is the middle of a, v, b where $a, b, v \in A_C$ and $A_C \setminus \{a, v, b\}$ is an independent (possibly empty) set. Then D contains exactly one of a or b, say b, and so $D \setminus \{b\}$ is a cds of G_{uv}. In any case $\gamma_c(G_{uv}) < \gamma_c(G)$. □

4. (T)CDD-CRITICAL GRAPHS G WITH SMALL $\gamma_c(G)$

In [1], Burton and Sumner characterized (totally) domination dot-critical graphs having domination number equal to 2.

Theorem 4.1 ([1]). *Let G be a graph of order $n \geq 4$ with $\gamma(G) = 2$. Then G is domination dot-critical if and only if the complement graph $\overline{G}$ of G is not complete and every component of $\overline{G}$ is a corona or a complete graph K_p, $p \geq 2$.*

Theorem 4.2 ([1]). *Let G be a graph of order $n \geq 2$ with $\gamma(G) = 2$. Then G is totally domination dot-critical if and only if every component of $\overline{G}$ is a corona.*

Theorem 4.3. *Let G be a connected graph with $\gamma_c(G) = 2$. Then G is connected domination dot-critical if and only if G is domination dot-critical.*

Proof. Assume that G is cdd-critical. Since $\gamma_c(G) = 2$, no vertex of G dominates all vertices and so $2 \leq \gamma(G) \leq \gamma_c(G) = 2$, i.e., $\gamma(G) = 2$. Now since G is cdd-critical, contracting any edge uv decreases $\gamma_c(G)$ and so $\gamma(G_{uv}) = 1$. Thus G is domination dot-critical.

Conversely, since $\gamma(G_{uv}) = 1$ for any edge uv, it follows that $\gamma_c(G_{uv}) = 1$ and so G is a cdd-critical graph. □

Theorem 4.4. *Let G be a connected graph with $\gamma_c(G) = 2$. Then G is totally connected domination dot-critical if and only if G is totally domination dot-critical.*

Proof. Assume that G is tcdd-critical. Then $\gamma(G) = 2$ since G has no vertex of degree $n - 1$. Also since the contraction of any edge or the contraction of any two vertices decrease $\gamma_c(G)$, we have $\gamma(G_{uv}) = 1$. Thus G is totally domination dot-critical.

Conversely, since $\gamma(G_{uv}) = 1$ for any vertices u and v, then $\gamma_c(G_{uv}) = 1$ and so G is a tcdd-critical graph. □

Triangle-free graphs. Here we characterize the connected triangle-free graphs with $\gamma_c(G) = 3$ that are connected domination dot-critical.

Let $\mathcal{F}_1$ be the family of graphs obtained from a cycle C_5 by duplicating every vertex arbitrarily many times (duplicating a vertex x means creating a vertex x' adjacent to every neighbor of x).

Let $\mathcal{F}_2$ be the family of graphs whose vertex-set can be partitioned into seven non-empty independent sets $\{a\}$, $\{b\}$, W, A_1, A_2, B_1, B_2, such that a is adjacent to all of $W \cup A_1 \cup A_2$, b is adjacent to all of $W \cup B_1 \cup B_2$, A_2 is adjacent to all of $B_1 \cup B_2$, B_2 is adjacent to all of $A_1 \cup A_2$, and there are no other edges.

Let $\mathcal{F}_3$ be the family of bipartite graphs G whose vertex-set can be partitioned into sets $\{a, b, a_1, b_1\}, W, X$ such that, for some $w \in W$, we have:

- a_1-a-w-b-b_1 is an induced P_5 in G;
- $W = N(a) \cap N(b)$;

- w is adjacent to every vertex of X;
- Every vertex of X has at least two neighbors and one non-neighbor in $\{a_1, b_1\} \cup W$;
- For every vertex $w' \in W$, either w' is adjacent to all of X, or there is a vertex $x(w') \in X$ that is adjacent to all of $\{a_1, b_1\} \cup W \setminus \{w'\}$ and not to w'.

Theorem 4.5. *Let G be a connected triangle-free graph with $\gamma_c(G) = 3$. Then G is connected domination dot-critical if and only if $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.*

Proof. We first prove that every member G of $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ is a connected triangle-free graph with $\gamma_c(G) = 3$ and that contracting any edge uv of G yields a graph G_{uv} with $\gamma_c(G_{uv}) = 2$.

Suppose that G is in $\mathcal{F}_1$. Then it is easy to check that $\gamma_c(G) = 3$ and that, for any edge uv of G, the graph G_{uv} has a connected dominating set of size two (which consists of the contracted vertex $\overline{uv}$ plus some neighbor of this vertex).

Now suppose that G is in $\mathcal{F}_2$ with the above notation. Clearly $\{a, w, b\}$ is a connected dominating set, and it is easy to see that G has no connected dominating set of size two. So $\gamma_c(G) = 3$. Now consider any edge uv of G. Pick arbitrary vertices $w \in W$, $a_i \in A_i$, $b_i \in B_i$, $i = 1, 2$. Up to isomorphism uv is one of the following three types:

(1) $uv = aw$; then $\{\overline{aw}, a_2\}$ is a connected dominating set of G_{uv}.
(2) $uv = aa'$, where $a' \in A_1 \cup A_2$; then $\{b, b_2\}$ is a connected dominating set of G_{uv}.
(3) $uv = a_2b'$, where $b' \in B_1 \cup B_2$; then $\{\overline{a_2b'}, a\}$ is a connected dominating set of G_{uv}.

Now suppose that G is in $\mathcal{F}_3$ with the above notation. Clearly $\{a, w, b\}$ is a connected dominating set, and it is easy to see that G has no connected dominating set of size two. So $\gamma_c(G) = 3$. Now consider any edge uv of G. Up to isomorphism uv is one of the following five types:

(1) $uv = a_1x$, where $x \in \{a\} \cup X$; then $\{w, b\}$ is a connected dominating set of G_{uv}.
(2) $uv = aw$; then $\{\overline{aw}, b\}$ is a connected dominating set of G_{uv}.
(3) $uv = aw'$, where $w' \in W$ and w' has a non-neighbor in X; then $\{w, x(w')\}$ is a connected dominating set of G_{uv}.
(4) $uv = wy$, where $y \in X$; then $\{\overline{wy}, a\}$ is a connected dominating set of G_{uv}.
(5) $uv = w'y$, where $y \in X$, $w' \in W$ and w' has a non-neighbor in X; then $\{w, x(w')\}$ is a connected dominating set of G_{uv}.

So in either case, the contraction of any edge of G gives a graph with $\gamma_c = 2$.

Now let us prove the converse part of the theorem. Let G be a connected triangle-free graph with $\gamma_c(G) = 3$, and suppose that G is connected domination dot-critical.

Given any graph H and subgraph S of H, we say that a vertex g of $H \setminus S$ is a *good neighbor* of S if g is adjacent to at least three vertices of S and $N(g) \cap S$ does not induce a path P_3. We make a few remarks.

(i) In G, any induced C_5 has no good neighbor.

(ii) In a graph G' such that $\gamma_c(G') = 2$ and any two triangles of G' intersect, any induced C_5 must have a good neighbor, and every $\gamma_c(G')$-set contains a good neighbor of every C_5.

(iii) In G, there is no subgraph with seven vertices $x_1, \ldots, x_5, y, z$ where $x_1, \ldots, x_5$ induce a C_5 in this order, y, z are adjacent, y is adjacent to x_1, z is adjacent to x_2 and there is no other edge between y, z and the x_i's (we call such a graph a θ_1).

(iv) In G, there is no subgraph with seven vertices $x_1, \ldots, x_6, z$ where $x_1, \ldots, x_6$ induce a C_6 in this order and z is adjacent to x_1 and x_4 (we call such a graph a θ_2).

(v) In G, there is no subgraph with seven vertices $x_1, \ldots, x_5, y, z$ where $x_1, \ldots, x_5$ induce a C_5 in this order, y is adjacent to x_1 and x_4, and z is adjacent to y and optionally to x_5 (we call such a graph a θ_3).

Proof of remarks. (i): If any C_5 has a good neighbor, then G contains a triangle.

(ii): Suppose that C is an induced C_5 in G' and $\{u, v\}$ is a connected dominating set of G'. Clearly u, v are not both in C. If u is in C and v is not, then v must be adjacent to the two non-neighbors of u in C and to u, so v is a good neighbor of C. If both u, v are not in C, then it is easy to see that either one of u, v is a good neighbor of C or (up to symmetry) u is adjacent to three consecutive vertices of C and v is adjacent to the other two, and then G' has two triangles that do not intersect.

(iii): For otherwise, vertices $x_1, \ldots, x_5$ induce in G_{yz} a C_5 that has no good neighbor (because it has no good neighbor in G by (i) and the contracted vertex is also not a good neighbor), which contradicts (ii) for $G' = G_{yz}$ (note that in G_{yz} every triangle contains the contracted vertex $\overline{yz}$).

(iv): For otherwise, vertices $x_1, \ldots, x_4, z$ induce in $G_{x_5x_6}$ a C_5 that has no good neighbor (because it has no good neighbor in G and the contracted vertex is also not a good neighbor), which contradicts (ii) for $G' = G_{x_5x_6}$.

(v): For otherwise, vertices $x_1, \ldots, x_5$ induce in G_{yz} a C_5 that has no good neighbor (because it has no good neighbor in G and the contracted vertex is also not a good neighbor), which contradicts (ii) for $G' = G_{yz}$. □

Now let $D = \{a, w, b\}$ be a $\gamma_c(G)$-set of G. Since G is triangle-free, D induces a path and we may assume up to symmetry that it is a-w-b. Since $\{w, b\}$ is not a dominating set, there is a non-empty set A of vertices adjacent to a and not to w or b, and there is a non-empty set B of vertices adjacent to b and not to w or a. Let $W = N(a) \cap N(b)$ be the set of common neighbors of a and b, and let $X = N(w) \setminus \{a, b\}$. Since G is triangle-free, A, B, W and X are disjoint sets, and since D is a dominating set we have $V(G) = D \cup A \cup B \cup W \cup X$.

We note that there cannot exist an induced $2K_2$ with vertices $u, u' \in A$, $v, v' \in B$ and edges $uv, u'v'$, for otherwise a, w, b, u, u', v, v' would induce a θ_2. The fact that there is no such $2K_2$ means that for any two vertices $u, u' \in A$ one of $N(u) \cap B$ and $N(u') \cap B$ is included in the other; and so this inclusion relation induces a total order $<$ on the vertices of A. A similar statement holds for the vertices of B. More formally, let A' be the set of vertices of A that have a neighbor in B, and define B' similarly. Then, using only the fact that there is no such $2K_2$, a classical result (see [4]) states that: If A' and B' are not empty, then for some integer $h \geq 1$ there exists a partition of A' into non-empty sets $A_1, \dots, A_h$ and a partition of B' into non-empty set $B_1, \dots, B_h$, such that any two vertices $u \in A_i, v \in B_j$ are adjacent if and only if $i + j \geq h + 1$. Let $A_0 = A \setminus A'$ and $B_0 = B \setminus B'$. If A' and B' are empty, put $h = 0$. So h is always defined. We may assume that we choose the set $D = \{a, w, b\}$ so that the corresponding number h is as large as possible, and, if $h = 0$, so that the set $W = N(a) \cap N(b)$ is as large as possible.

If $h > 0$, for each $i \in \{1, \dots, h\}$ we pick arbitrary vertices $a_i \in A_i$ and $b_i \in B_i$. We claim that:

(4.1) $$\text{Either } A_0 = \varnothing \text{ or } |B'| \leq 1.$$

For suppose on the contrary that there are vertices $a_0 \in A_0$ and $b', b'' \in B'$. Thus we have $h \geq 1$, so there is a vertex $a_h \in A_h$, and a_h is adjacent to both b', b''. Let $G' = G_{aa_0}$, let c be the contracted vertex $\overline{aa_0}$ in G', and let D' be a connected dominating set of size 2 of G'. In G', vertices a_h, b', b, w, c induce a C_5, so, by (iii), D' must contain a good neighbor x of that C_5. Since G is triangle-free, there are only two possibilities for x:

- x is adjacent to c, a_h, b. Then, in G, vertex x is adjacent to a_0, a_h, b (and not to a, w, b', b''); so $x \in B$ and a_0 has a neighbor in B, a contradiction.
- x is adjacent to c, w, b'. Then, in G, vertex x is adjacent to a_0, w, b' (and not to a, b, a_h), and then either $x, a, b, w, a_0, a_h, b''$ induce a θ_1 (if x is not adjacent to b'') or $x, a, b, a_0, a_h, b', b''$ induce a θ_3 (if x is adjacent to b''), a contradiction.

Thus (4.1) holds.

Similarly, we have:

(4.2) $$\text{Either } B_0 = \varnothing \text{ or } |A'| \leq 1.$$

Next, we claim that:

(4.3) $$h \leq 2.$$

For suppose that $h \geq 3$. So there are edges

$$a_h b_h, a_h b_{h-1}, a_h b_1, a_{h-1} b_h, a_{h-1} b_{h-1}, a_1 b_h$$

and no other edges between these six vertices. Let $G' = G_{a_{h-1}b_{h-1}}$, let c be the contracted vertex $\overline{a_{h-1}b_{h-1}}$ in G', and let D' be a connected dominating set of size 2 of G'. In G', vertices a_h, b_h, b, w, a induce a C_5 whose only good neighbor is c, so, by (ii), we have $D' = \{c, x\}$ for some x. In G', vertex c is not adjacent to a_1, b_1, w, so x must be adjacent to them and to c. So, in G, vertex x is adjacent to a_1, b_1, w (and not to a, b) and to one of a_{h-1}, b_{h-1}, but not to both since G contains no triangle. But then $x, a, b, a_1, b_1, a_{h-1}, b_{h-1}$ induce a θ_1, a contradiction. Thus (4.3) holds.

Now we distinguish among three cases.

Case 1: $h = 2$:

So there are edges a_2b_2, a_2b_1, a_1b_2, and a_1, b_1 are not adjacent. Since B contains two vertices b_1, b_2 that have a neighbor in A, by (4.1), we have $A_0 = \varnothing$. Similarly $B_0 = \varnothing$. Suppose that $X \neq \varnothing$ and let x be any vertex in X. If x is adjacent to a_2, then $x, a_2, b_1, b, w, a, a_1$ induce a θ_3, a contradiction. Thus, and by symmetry, x has no neighbor in $A_2 \cup B_2$. Let $G' = G_{wx}$, let c be the contracted vertex $\overline{wx}$ in G', and let D' be a connected dominating set of size 2 of G'. In G', vertices a_2, b_2, b, a, c induce a C_5, so, by (ii), D' must contain a good neighbor z of that C_5. Since G is triangle-free, and up to symmetry, we may assume that z is adjacent to a, b_2, c. Then, in G, vertex z is adjacent to a, b_2, x (and not to w, b or any vertex of A), so z is in A. Since x has no neighbor in A_2, we have $z \in A_1$. Then x is adjacent to every $b_1 \in B_1$, for otherwise a, w, b, a_2, b_1, z, x induce a θ_1. So x has a neighbor in B_1, and then, by a symmetric argument, x is also adjacent to every vertex of A_1. Now let $G'' = G_{aa_1}$, let d be the contracted vertex $\overline{aa_1}$ in G'', and let D'' be a connected dominating set of size 2 of G''. In G'', vertices b, w, d, a_2, b_1 induce a C_5, so, by (ii), D'' must contain some good neighbor u of that C_5. Put $D'' = \{u, v\}$. It is easy to see that the set of good neighbors of that C_5 is exactly $X \cup B_2$. If $u \in X$, then u is not adjacent to a_2, b_2, so v should be adjacent to these two, but then v, a_2, b_2 would induce a triangle in G. If $u \in B_2$, then u is not adjacent to b_1, x, so v should be adjacent to these two, but then v, b_1, x would induce a triangle in G. Thus we always find a contradiction. So $X = \varnothing$, and then G is in $\mathcal{F}_2$.

Case 2: $h = 1$:

Consider any $x \in X$. If x has no neighbor in $A_1 \cup B_1$, then in G_{wx} vertices $a, b, \overline{wx}, a_1, b_1$ induce a C_5, and it is easy to see that this C_5 has no good neighbor, a contradiction. So every vertex of X has a neighbor in A_1 or in B_1, and not in both since G is triangle-free. Let X_A be the set of vertices of X that have a neighbor in A_1, and define X_B similarly. Thus $X = X_A \cup X_B$. There is no $2K_2$ with vertices

$x, x' \in X$, $u, u' \in A_1$ and edges $xu, x'u'$, for otherwise x, x', u, u', b_1, b, w would induce a θ_2. It follows that there is a vertex $u_1 \in A_1$ that is adjacent to all of X_A. Similarly, there is no $2K_2$ between X_B and B_1, and there is a vertex $v_1 \in B_1$ that is adjacent to all of X_B. Suppose that there is a vertex $a_0 \in A_0$. Then, in G_{aa_0}, vertices $\overline{aa_0}, w, b, u_1, v_1$ induce a C_5, which must have a good neighbor x, and it is easy to see that x must be adjacent (in G) to w, v_1, a_0. Suppose that there is also a vertex $b_0 \in B_0$. Then, similarly, there is a vertex y adjacent to w, u_1, b_0. But then $w, x, y, u_1, v_1, a, b_0$ induce a θ_2, a contradiction. So $B_0 = \varnothing$. Now we observe that the set $D^* = \{a, u_1, v_1\}$ is a connected dominating set of G (by the definition of u_1, v_1 and since $B_0 = \varnothing$), and, because of vertices w, a_0, x, b, we see that D^* satisfies the situation described in Case 1 (i.e., the "h" of D^* is equal to 2), which contradicts the choice of D. So $A_0 = \varnothing$ and similarly $B_0 = \varnothing$. Thus $A = A_1$ and $B = B_1$. Now we observe that the set $D^{**} = \{w, b, v_1\}$ is a connected dominating set of G, so, if any $x \in X_A$ has a non-neighbor $a_1 \in A_1$, then, because of a, x, u_1, a_1, the set D^{**} satisfies the situation of Case 1 (i.e., the "h" of D^{**} is equal to 2), again a contradiction. So we may assume that every vertex of X_A is adjacent to every vertex of A, and similarly every vertex of X_B is adjacent to every vertex of B. The set $\{b, v_1, u_1\}$ is a dominating set of G, so, by a similar argument, every vertex of X_A is adjacent to every vertex of W; and similarly every vertex of X_B is adjacent to every vertex of W. Thus G is in $\mathcal{F}_1$.

Case 3: $h = 0$:

So there is no edge between A and B, and G is a bipartite graph, where $A \cup B \cup W$ and $X \cup \{a, b\}$ are two stable sets that form a bipartition of $V(G)$. We make a new remark:

(vi) Let u, v be two adjacent vertices of G, and suppose that the graph G_{uv} contains a P_5 p_1-p_2-p_3-p_4-p_5. Let D be any connected dominating set of G_{uv}; then either D contains the contracted vertex $\overline{uv}$ or one vertex of D is adjacent to p_1, p_3, p_5 and the other is adjacent to p_2, p_4.

Indeed, if D does not contain the contracted vertex $\overline{uv}$, then D is a connected dominating set of the bipartite graph induced by $D \cup \{p_1, \dots, p_5\}$, so it must be that one vertex of D is adjacent to p_1, p_3, p_5 and the other is adjacent to p_2, p_4.

Now we claim that $|A| = 1$. For suppose that A has at least two elements. Pick any $a_1 \in A$ and consider the graph G_{aa_1}, where we call c the contracted vertex $\overline{aa_1}$. This graph has a dominating set D of size 2. Note that a'-c-w'-b-b' induces a P_5 in G_{aa_1} for every $a' \in A \setminus \{a_1\}$, $w' \in W$, and $b' \in B$. The set D cannot contain c, for otherwise the other vertex y of D should be adjacent to b and b' and then G would contain the triangle $\{y, b, b'\}$. So, by (v), D contains a vertex x that is adjacent to every element of $(A \setminus \{a_1\}) \cup W \cup B$. It follows that $D' = \{a, w, x\}$ is a connected dominating set of G, and the "h" of D' is equal to 0 since we are not in cases 1 or 2. We have $N(a) \cap N(x) = W \cup (A \setminus \{a_1\})$, so $|N(a) \cap N(x)| > |N(a) \cap N(b)|$, which contradicts the choice of D when $h = 0$. Thus $|A| = 1$ and similarly $|B| = 1$. Put $A = \{a_1\}$ and $B = \{b_1\}$.

We observe that every vertex $x \in X$ has a non-neighbor in $\{a_1, b_1\} \cup W$, for otherwise $\{x, w\}$ would be a connected dominating set of size two of G. Moreover, we claim that every vertex $x \in X$ has at least two neighbors in $\{a_1, b_1\} \cup W$. For suppose that w is the only neighbor of x in W. Consider the graph G_{xw}, and let D be a connected dominating set of G_{xw}. Note that a_1-a-w'-b-b_1 induces a P_5 in G_{xw} for every $w' \in W$. So, by (v), there is a vertex y that is adjacent to a_1 and b_1 and to every vertex of $W \setminus \{w\}$. Clearly y is in X, so it is also adjacent to w. But then $\{w, y\}$ is a connected dominating set of G, a contradiction.

Finally we claim that for every vertex $w' \in W$, either w' is adjacent to all of X, or there is a vertex $x(w') \in X$ that is adjacent to all of $\{a_1, b_1\} \cup W \setminus \{w'\}$ (and not to w'). For suppose that w' has a non-neighbor u in X. Consider the graph $G_{w'b}$, and let $D = \{d, d'\}$ be a connected dominating set of $G_{w'b}$. Call c the contracted vertex $\overline{w'b}$. Note that c is not in D, for otherwise the other vertex of D should be adjacent to a_1 and u, which is impossible since G is bipartite. Since d, d' are adjacent, one of them, say d', is not in X, and so (since D is a dominating set of $G_{w'b}$) d must be adjacent to a_1, b_1 and all of $W \setminus \{w'\}$. Thus d can play the role of the desired vertex $x(w')$.

It follows from the preceding points that G is in class $\mathcal{F}_3$. This completes the proof. □

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