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$\{-1,2\}$-HYPOMORPHY AND HEREDITARY HYPOMORPHY COINCIDE FOR POSETS

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Abstract. Let P and P' be two finite posets on the same vertex set V. The posets P and P' are *hereditarily hypomorphic* if for every subset X of V, the induced subposets $P(X)$ and $P'(X)$ are isomorphic. The posets P and P' are $\{-1,2\}$-*hypomorphic* if for every subset X of V with $|X| \in \{2, |V|-1\}$, the subposets $P(X)$ and $P'(X)$ are isomorphic. P. Ille and J. X. Rampon [7] show that if two posets P and P', with at least 4 vertices, are $\{-1,2\}$-hypomorphic, then P and P' are isomorphic. Under the same hypothesis, we prove that P and P' are hereditarily hypomorphic. Moreover, we characterize the pairs of hereditarily hypomorphic posets.

1. Introduction

A *partially ordered set* (or *poset*) P is an ordered pair $(V(P), E(P))$, where $V(P)$ is a finite set, called the *vertex set* of P, and $E(P)$ is an irreflexive, antisymmetric and transitive binary relation on $V(P)$. By $x <_P y$, we denote the fact that $(x, y) \in E(P)$. The *comparability graph* of P is the graph $G(P) = (V(P), E(G(P)))$, where for $x, y \in V(P)$, $\{x, y\} \in E(G(P))$ if and only if either $x <_P y$ or $y <_P x$. If $G(P)$ is complete (resp. empty), the poset P is called a *total order* (resp. an *empty order*). Given $x \neq y$ in $V(P)$, if $\{x, y\} \notin E(G(P))$, we set $x \parallel_P y$. The *complement* of $G(P)$ is the graph $\overline{G(P)} := (V(P), [V(P)]^2 \setminus E(G(P)))$ where $[V(P)]^2$ denotes the set of pairs of distinct elements of $V(P)$. Let $X \subseteq V(P)$, *the subposet* of P *induced by* X is the poset $(X, E(P) \cap (X \times X))$, denoted $P(X)$. For convenience, if $x \in V(P)$, the subposet $P(V(P) \setminus \{x\})$ is denoted $P - x$. The *dual* P^* of P is the poset $(V(P), E(P^*))$, where for $x, y \in V(P)$, $(x, y) \in E(P^*)$ if and only if $(y, x) \in E(P)$. A subset X of $V(P)$ is an *interval* of P whenever for every $y \in V(P) \setminus X$, either for all $x \in X$, $x <_P y$, or for all $x \in X$, $y <_P x$, or for all $x \in X$, $x \parallel_P y$. Clearly, the empty set, $V(P)$ and the singletons of $V(P)$ are intervals of P, called *trivial intervals.* The poset P is said *indecomposable* whenever all its intervals are trivial; otherwise P is said

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decomposable. A subset X of $V(P)$ is a *strong interval* of P [5, 10], provided that X is an interval of P satisfying the following: for any interval Y of P, if $X \cap Y \neq \varnothing$, then $X \subseteq Y$ or $Y \subseteq X$. In what follows, $\mathcal{S}(P)$ denotes the family of strong intervals of P fulfilling: $X \neq V(P)$ and for every strong interval Y of P, if $X \subseteq Y$, then $Y = X$ or $Y = V(P)$. A partition $\mathcal{S}$ of $V(P)$, all of the elements of which are intervals of P, is called an *interval partition* of P. For such a partition, define the *quotient* $P/\mathcal{S} = (\mathcal{S}, E(P/\mathcal{S}))$ of P by $\mathcal{S}$ as follows: for $X \neq Y$ in $\mathcal{S}$, $(X,Y) \in E(P/\mathcal{S})$ if and only if $(x,y) \in E(P)$, where $x \in X$ and $y \in Y$. The inverse operation of the quotient is the *lexicographical sum* defined as follows: let P be a poset, with any $x \in V(P)$ is associated a poset P_x so that for $x \neq y$ in $V(P)$, $V(P_x) \cap V(P_y) = \varnothing$. The lexicographical sum of the P_x's over P is the poset

$$P(P_x; x \in V(P)) := \left(\bigcup_{x \in V(P)} V(P_x), E\left(P(P_x; x \in V(P))\right) \right)$$

defined in the following manner: given $a \neq b$ in $\cup_{x \in V(P)} V(P_x)$, $(a,b) \in E(P(P_x; x \in V(P)))$ provided that either $x = y$ and $(a,b) \in E(P_x)$, or $x \neq y$ and $(x,y) \in E(P)$, where x and y are the vertices of P such that $a \in V(P_x)$ and $b \in V(P_y)$. This poset replaces each vertex x of P by P_x in such a way that $V(P_x)$ becomes an interval. We say that the vertex x of P is *dilated by* P_x. Let $P := (V(P), E(P))$ and $P' := (V(P'), E(P'))$ be two posets. A one-to-one correspondence f from $V(P)$ onto $V(P')$ is an *isomorphism* from P onto P' provided that for $x, y \in V(P)$, $(x,y) \in E(P)$ if and only if $(f(x), f(y)) \in E(P')$. The posets P and P' are *isomorphic*, which is denoted $P \simeq P'$, if there is an isomorphism from P onto P'.

Following Ulam's reconstruction for graphs [15, 2], a similar conjecture for posets [14, 12] asserts that two posets P and P' on the same vertex set V of $n \geq 4$ vertices are isomorphic provided that the induced subposets $P(X)$ and $P'(X)$ of P and P' to the $(n-1)$-element subsets of V are isomorphic. The cardinality of a set X, will be denoted by $\mid X \mid$. Let P and P' be two posets on a same vertex set V, let $n :=\mid V \mid$. Let k be a non-negative integer with $k < n$. The posets P and P' are $\{k\}$-*hypomorphic*, whenever for every subset X of V of cardinality k, $P'(X)$ and $P(X)$ are isomorphic. The posets P and P' are $\{-k\}$-*hypomorphic* if they are $\{n-k\}$-hypomorphic. Let F be a set of integers such that for every $k \in F$, $0 < |k| < n$. The posets P and P' are F-*hypomorphic* whenever for every $k \in F$, P' and P are $\{k\}$-hypomorphic. Thus, two posets P and P' are $\{-1,2\}$-*hypomorphic* if and only if $G(P) = G(P')$ and for all $x \in V$, $P - x$ and $P' - x$ are isomorphic. The poset P is F-*reconstructible* if every poset that is F-hypomorphic to P is isomorphic to P. The F-reconstruction generalizes Ulam's reconstruction [15, 2]. The posets P and P' are *hereditarily hypomorphic* if for all $X \subseteq V$, $P(X)$ and $P'(X)$ are isomorphic.

P. Ille and J. X. Rampon [7] show that if two posets P and P', with $|V| \geq 4$, are $\{-1,2\}$-hypomorphic, then P and P' are isomorphic. Under the same hypothesis, we prove that P and P' are hereditarily hypomorphic (Theorem 2.4). Moreover, we characterize the pairs of hereditarily hypomorphic posets (Theorem 2.14). Note that K.B. Reid and C. Thomassen studied the notion of hereditary hypomorphy in the case of tournaments [13], called by them "hereditarily isomorphy."

2. The $\{-1,2\}$-hypomorphy

First, recall that the $\{2,3\}$-hypomorphy was studied by J.G. Hagendorf [6], where he proved the following result.

Theorem 2.1 ([6]). *Every poset with at least* 4 *vertices is* $\{2,3\}$*-reconstructible.*

The following corollary is a consequence of Theorem 2.1.

Corollary 2.2. *Given two posets P and P' on the same vertex set V, with $|V| \geq 4$, then P and P' are hereditarily hypomorphic if and only if P and P' are $\{2,3\}$-hypomorphic.*

The $\{-1,2\}$-hypomorphy of posets was studied by P. Ille and J. X. Rampon [7]. They show the following result.

Theorem 2.3 ([7]). *Let P and P' be two posets on the same vertex set V, with $|V| \geq 4$. If P and P' are $\{-1,2\}$-hypomorphic then P and P' are isomorphic.*

We improve Theorem 2.3 with the following result.

Theorem 2.4. *Let P and P' be two posets on the same vertex set V, with $|V| \geq 4$. Then P and P' are $\{-1,2\}$-hypomorphic if and only if P and P' are hereditarily hypomorphic.*

In the course of the proof of Theorem 2.3, the authors [7] establish the following proposition, which will be useful for the proof of Theorem 2.4.

Proposition 2.5. *Let P and P' be two $\{-1,2\}$-hypomorphic posets with $|V(P)| \geq 5$. If $P := Q(P_x; x \in V(Q))$, where Q is indecomposable and $|V(Q)| \geq 4$, then $P' = Q(P'_x; x \in V(Q))$, and for every $x \in V(Q)$, $V(P_x) = V(P'_x)$.*

The proof of Theorem 2.4 needs the following results which describe decomposition of posets.

Theorem 2.6 ([5, 10]). *Let P be a poset, with $|V(P)| \geq 2$. Then one of the following conditions is satisfied:*

(i) *$G(P)$ is disconnected, $\mathcal{S}(P)$ is the interval partition of P consisting of the connected components of $G(P)$, and $P/\mathcal{S}(P)$ is empty.*
(ii) *$\overline{G(P)}$ is disconnected, $\mathcal{S}(P)$ is the interval partition of P consisting of the connected components of $\overline{G(P)}$, and $P/\mathcal{S}(P)$ is a total order.*

(iii) *$G(P)$ and $\overline{G(P)}$ are connected, $|\mathcal{S}(P)| \geq 4$, and $P/\mathcal{S}(P)$ is indecomposable.*

Theorem 2.7 ([5, 10])**.** *Given posets P and P', if $G(P) = G(P')$ and if P is indecomposable, then $P' = P$ or $P' = P^*$.*

Proposition 2.8 ([5, 10])**.** *Given posets P and P', if $G(P) = G(P')$, then P and P' have the same strong intervals and, consequently, $\mathcal{S}(P) = \mathcal{S}(P')$.*

We continue with some of the lemmas in reconstruction.

Let P and H be posets with $|V(H)| < |V(P)|$ and $x \in V(P)$. We use the notation $n(P, H, x) :=| \{F \subseteq V(P), x \in F \text{ and } P(F) \simeq H\} |$.

Lemma 2.9 ([8])**.** *Let P, P' and H be posets. If P and P' are $\{-1\}$-hypomorphic and if $|V(H)| < |V(P)|$, then for every $x \in V(P)$, $n(P, H, x) = n(P', H, x)$.*

Lemma 2.10 ([11])**.** *Let P and P' be posets. If P and P' are $\{q\}$-hypomorphic where $1 \leq q \leq |V(P)| - 1$, then for $p = 1, \ldots, \min(q, |V(P)| - q)$, P and P' are $\{p\}$-hypomorphic.*

Lemma 2.11 ([1, 7])**.** *Let $P(P_x; x \in V(P))$ and $Q(Q_y; y \in V(Q))$ be lexicographical sums fulfilling the following conditions:*

(i) *$P(P_x; x \in V(P))$ and $Q(Q_y; y \in V(Q))$ are $\{-1\}$-hypomorphic.*
(ii) *P and Q are indecomposable with at least 4 vertices.*
(iii) *$|\{x \in V(P) \ : \ |V(P_x)| \geq 2\}| \geq 2$ and*
$|\{y \in V(Q) \ : \ |V(Q_y)| \geq 2\}| \geq 2$.

Then P and Q are isomorphic, and for all $u \in \cup_{x \in V(P)} V(P_x)$, $P_\alpha \simeq Q_\beta$, where $\alpha \in V(P)$, $\beta \in V(Q)$, and $u \in V(P_\alpha) \cap V(Q_\beta)$.

The following lemma is a personal communication of A. Boussaïri [3].

Lemma 2.12. *Let $p \geq 2$ be an integer, $i \in \{1, \ldots, p\}$, R be a poset with vertex set $\{1, \ldots, p\}$ and H (resp. H') a poset such that $V(H)$ (resp. $V(H')$) is disjoint from $\{1, \ldots, p\}$. Let G (resp. G') be the poset obtained from R by dilating the vertex i by H (resp. H'). Then $G \simeq G'$ if and only if $H \simeq H'$.*

This lemma is generalized as follows.

Lemma 2.13. *Let $p \geq 2$ be an integer, $i \in \{1, \ldots, p\}$, R and R' be two isomorphic posets with the same vertex set $\{1, \ldots, p\}$, f an isomorphism from R onto R', and H (resp. H') a poset such that $V(H)$ (resp. $V(H')$) is disjoint from $\{1, \ldots, p\}$. Let G (resp. G') be the poset obtained from R (resp. R') by dilating the vertex i (resp. $f(i)$) by H (resp. H'). Then $G \simeq G'$ if and only if $H \simeq H'$.*

Proof. If $H \simeq H'$, then the conclusion is trivial. Assume that $G \simeq G'$. Let G'' be the poset obtained from R by dilating the vertex i by H'. It is clear that $G' \simeq G''$; thus $G \simeq G''$ and we conclude using Lemma 2.12. □

Proof of Theorem 2.4. The sufficient condition is obvious. The proof of the necessary condition is done by induction on $n = |V|$. If $n = 4$, then P and P' are $\{2,3\}$-hypomorphic, and thus we conclude using Theorem 2.1. Assume that $n \geq 5$ and that the assertion is true for every pair of posets $\{L, L'\}$ with $4 \leq |V(L)| < n$. Note that, by Theorem 2.3, $P \simeq P'$. We shall study the three cases of Gallai's decomposition of P.

CASE 1: $G(P)$ is disconnected, $\mathcal{S}(P)$ is the interval partition of P consisting of the connected components of $G(P)$, and $P/\mathcal{S}(P)$ is empty.

By Proposition 2.8, $\mathcal{S}(P) = \mathcal{S}(P')$. By denoting $Q := P/\mathcal{S}(P)$, we get $P = Q(P_x; x \in V(Q))$ and $P' = Q(P'_x; x \in V(Q))$, where Q is an empty graph and for all $x \in V(Q)$, $V(P_x) = V(P'_x) \in \mathcal{S}(P)$. To conclude, it is sufficient to prove that for all $x \in V(Q)$, the posets P_x and P'_x are hereditarily hypomorphic.

First, we shall prove that $P_x \simeq P'_x$ for all $x \in V(Q)$. Let $x \in V(Q)$, and let $a \in V(P_x)$. By Lemma 2.9, $n(P, P_x, a) = n(P', P_x, a)$. Clearly, for all $A \subseteq V$, if there are $y_1 \neq y_2$ in $V(Q)$ such that

$$A \cap V(P_{y_1}) \neq \varnothing \text{ and } A \cap V(P_{y_2}) \neq \varnothing,$$

then $G(P(A))$ is disconnected, and thus $P(A)$ and P_x are non isomorphic. It ensues that $n(P, P_x, a) = 1$, and then $n(P', P_x, a) = 1$. From this we get $P_x \simeq P'_x$.

Now let $x \in V(Q)$. If $|V(P_x)| \leq 3$, then P_x and P'_x are hereditarily hypomorphic. Assume that $|V(P_x)| \geq 4$ and let $u \in V(P_x)$. By hypothesis $P - u$ and $P' - u$ are isomorphic. Consider an element y of $V(P_x)$, and let

$$P_1 := P((V(P) \setminus V(P_x)) \cup \{y\})$$
$$\big(\text{resp. } P'_1 := P'((V(P) \setminus V(P_x)) \cup \{y\})\big).$$

The poset $P - u$ (resp. $P' - u$) is obtained from P_1 (resp. P'_1) by dilating the vertex y by $P_x - u$ (resp. $P'_x - u$). As $P'_z \simeq P_z$ for all $z \in V(Q) \setminus \{x\}$, then P_1 and P'_1 are isomorphic by an isomorphism f satisfying $f(y) = y$. Since $P - u \simeq P' - u$, by Lemma 2.13, $P_x - u \simeq P'_x - u$. Thus the posets P_x and P'_x are $\{-1, 2\}$-hypomorphic, and by the induction hypothesis, P_x and P'_x are hereditarily hypomorphic.

CASE 2: $\overline{G(P)}$ is disconnected, $\mathcal{S}(P)$ is the interval partition of P consisting of the connected components of $\overline{G(P)}$, and $P/\mathcal{S}(P)$ is a total order.

If P is a total order, then P and P' are hereditarily hypomorphic. Assume that P is not a total order. From the partition $\mathcal{S}(P) = \mathcal{S}(P')$ of V, we define the following partition $\mathcal{S}_1(P)$ (resp. $\mathcal{S}_1(P')$) of V. From $A \subseteq V$, $A \in \mathcal{S}_1(P)$ (resp. $A \in \mathcal{S}_1(P')$) if and only if either $A \in \mathcal{S}(P)$ (resp. $A \in \mathcal{S}(P')$) and $|A| \geq 2$, or A is a maximal union of consecutive vertices of $P/\mathcal{S}(P)$ (resp. $P'/\mathcal{S}(P')$) which are singletons. Let $H := P/\mathcal{S}_1(P)$ (resp. $H' := P'/\mathcal{S}_1(P')$). Clearly, H and H' are total orders. Since $P' \simeq P$, it follows that $H' \simeq H$. Let Q be the usual order on

$\{1,\ldots,|\mathcal{S}_1(P)|\}$. Clearly $P = Q(P_x; x \in V(Q))$ and $P' = Q(P'_x; x \in V(Q))$ where $V(P_x) \in \mathcal{S}_1(P)$ and $V(P'_x) \in \mathcal{S}_1(P')$ for all $x \in V(Q)$. Since $P' \simeq P$, it follows that $P'_x \simeq P_x$ for all $x \in V(Q)$.

First, we shall prove that $V(P_x) = V(P'_x)$ for every $x \in V(Q)$. Suppose by contradiction that there is $x \in V(Q)$ such that $V(P_x) \neq V(P'_x)$, and choose x minimal in $V(Q)$ with respect to this property. Note that $x <_Q |\mathcal{S}_1(P)|$. Let $u \in V(P_x) \setminus V(P'_x)$. For all $y \in V(Q)$, either P_y is a total order or $\overline{G(P_y)}$ is connected.

If P_x is a total order, then clearly $P-u$ and $P'-u$ are not isomorphic, contradicting the fact that P and P' are $\{-1\}$-hypomorphic. If P_x is not a total order, as $P-u \simeq P'-u$, then by the choice of x, there is a $z \in V(P_{x+1})$ such that $P((V(P_x) \setminus \{u\}) \cup \{z\}) \simeq P'_x$. But this is impossible since $\overline{G(P((V(P_x) \setminus \{u\}) \cup \{z\}))}$ is disconnected and $\overline{G(P'_x)}$ is connected.

Now, it is sufficient to prove that, for all $x \in V(Q)$, P_x and P'_x are hereditarily hypomorphic. Let $x \in V(Q)$. If $|V(P_x)| \leq 3$, then we conclude using the facts that $P_x \simeq P'_x$, and P and P' are $\{2\}$-hypomorphic. Assume that $|V(P_x)| \geq 4$, and let $u \in V(P_x)$. By hypothesis, $P-u \simeq P'-u$. Consider an element y of $V(P_x)$, and let

$$P_1 := P((V(P) \setminus V(P_x)) \cup \{y\})$$
$$\big(\text{resp. } P'_1 := P'((V(P) \setminus V(P_x)) \cup \{y\})\big).$$

The poset $P-u$ (resp. $P'-u$) is obtained from P_1 (resp. P'_1) by dilating the vertex y by $P_x - u$ (resp. $P'_x - u$). As $P'_z \simeq P_z$ for all $z \in V(Q) \setminus \{x\}$, then P_1 and P'_1 are isomorphic by an isomorphism f satisfying $f(y) = y$. Since $P-u \simeq P'-u$, by Lemma 2.13, $P_x - u \simeq P'_x - u$. Thus the posets P_x and P'_x are $\{-1,2\}$-isomorphic, and by the induction hypothesis, P_x and P'_x are hereditarily hypomorphic.

Case 3: $G(P)$ and $\overline{G(P)}$ are connected, $|\mathcal{S}(P)| \geq 4$, and $P/\mathcal{S}(P)$ is indecomposable.

Denoting $Q := P/\mathcal{S}(P)$, we get $P = Q(P_x; x \in V(Q))$ and Q is indecomposable. By Proposition 2.5, $P' = Q(P'_x; x \in V(Q))$, and for every $x \in V(Q)$, $V(P_x) = V(P'_x)$. To conclude, it is sufficient to prove that for every $x \in V(Q)$, the posets P_x and P'_x are hereditarily hypomorphic. If for all $x \in Q$, $|V(P_x)| = 1$, then $P' = P$. Let us study the following two cases.

Case 3.1: $|\{x \in Q \ : \ |V(P_x)| \geq 2\}| \geq 2$.

By Lemma 2.11, for every $x \in V(Q)$, $P_x \simeq P'_x$. Let $x \in V(Q)$. If $|V(P_x)| \leq 3$, then P_x and P'_x are hereditarily hypomorphic. Assume that $|V(P_x)| \geq 4$ and let $u \in V(P_x)$. By hypothesis, $P-u \simeq P'-u$. Consider an element y of $V(P_x)$, and let $P_1 := P((V(P) \setminus V(P_x)) \cup \{y\})$ (resp. $P'_1 := P'((V(P) \setminus V(P_x)) \cup \{y\})$). The poset $P-u$ (resp. $P'-u$) is obtained from P_1 (resp. P'_1) by dilating the vertex y by $P_x - u$ (resp. $P'_x - u$). It is clear that P_1 and P'_1 are isomorphic by an isomorphism f satisfying $f(y) = y$. Since $P - u \simeq P' - u$, by Lemma 2.13, $P_x - u \simeq P'_x - u$. Thus the posets P_x and P'_x are $\{-1, 2\}$-hypomorphic, and by the induction hypothesis, P_x and P'_x are hereditarily hypomorphic.

CASE 3.2: $\{x \in Q \ : \ |V(P_x)| \geq 2\}$ is a singleton $\{x\}$.

If $|V(P_x)| = 2$, the conclusion is immediate. If $|V(P_x)| = 3$, the poset P (resp. P') is obtained from Q by dilating the vertex x by P_x (resp. P'_x). Since $P \simeq P'$, by Lemma 2.12, $P_x \simeq P'_x$; thus P_x and P'_x are hereditarily hypomorphic. If $|V(P_x)| \geq 4$, let $u \in V(P_x)$. By hypothesis there is an isomorphism f_u from $P - u$ onto $P' - u$. Necessarily, $f_u(V(P_x)\setminus\{u\}) = (V(P_x)\setminus\{u\})$, and then $P_x - u \simeq P'_x - u$. Thus P_x and P'_x are $\{-1, 2\}$-hypomorphic, and by the induction hypothesis, P_x and P'_x are hereditarily hypomorphic.

□

Theorem 2.14. *Let P and P' be two posets on the same vertex set of at least 4 elements. If P and P' are $\{2, 3\}$-hypomorphic, then there is a poset Q such that $P = Q(P_x; x \in V(Q))$ and $P' = Q(P'_x; x \in V(Q))$, where for every $x \in V(Q)$, the posets P_x and P'_x are two total orders with the same vertex set.*

Proof. Consider hereditarily hypomorphic posets P and P'. Given $x \neq y \in V(P)$, $\{x, y\}$ is called a *difference pair* if $P(\{x, y\}) \neq P'(\{x, y\})$. This notion is due to G. Lopez [9]. To localize the difference pairs, we use the strong intervals of P (and of P') or their decomposition tree as follows. Recall that for every strong interval X, we have $S(P(X)) = S(P'(X))$. Denote by $\mathcal{D}$ the family of the strong intervals X such that $P(X)/S(P(X)) \neq P'(X)/S(P(X))$. Let X be an element of $\mathcal{D}$. Applying Proposition 2.5 to $P(X)$ and $P'(X)$, $P(X)/S(P(X))$ and $P'(X)/S(P(X))$ are total orders. It follows from Case 2 of the proof of Theorem 2.4, where P and P' are replaced by $P(X)$ and $P'(X)$ respectively, that there exists an interval partition Q_X of $P(X)$ and of $P'(X)$ satisfying:

- The partition $S(P(X))$ is finer than Q_X, i.e., for each $X \in S(P(X))$, there is $Y \in Q_X$ such that $X \subseteq Y$.
- for every $Y \in S(P(X))$, if $|Y| > 1$, then $Y \in Q_X$;
- for every $Y \in Q_X \setminus S(P(X))$, $P(Y)$ and $P'(Y)$ are total orders;
- $P(X)/Q_X = P'(X)/Q_X$.

Now, consider the union $\mathcal{U}$ of the partial partitions $Q_X \setminus S(P(X))$ when X describes $\mathcal{D}$. It is easy to verify that the elements of $\mathcal{U}$ are mutually disjoint. If $\cup\mathcal{U} \neq V(P)$, we add to $\mathcal{U}$ the singletons $\{x\}$ whenever $x \in V(P) \setminus \cup\mathcal{U}$. In this way, we obtain an interval partition $\widetilde{\mathcal{U}}$ of P and of P'. Since for $X \in \mathcal{D}$, and for $Y \in Q_X \setminus S(P(X))$, $P(Y)$ is a total order, we have that $P(Z)$ is a total order for every $Z \in \widetilde{\mathcal{U}}$. Finally, to obtain Theorem 2.14, it suffices to verify that $P/\widetilde{\mathcal{U}} = P'/\widetilde{\mathcal{U}}$. Equivalently, given $x \neq x' \in V(P)$ such that $\{x, x'\}$ is a difference pair, we have to show that there is an element of $\widetilde{\mathcal{U}}$

which contains x and x'. Indeed, let X be the smallest strong interval such that $x, x' \in X$. There are $Y, Y' \in S(P(X))$ such that $x \in Y$ and $x' \in Y'$. By the minimality of X, we have $Y \neq Y'$. Therefore,

$$(P(X)/S(P(X)))(\{Y, Y'\}) \neq (P'(X)/S(P(X)))(\{Y, Y'\})$$

and thus $X \in \mathcal{D}$. Furthermore, as $P(X)/Q_X = P'(X)/Q_X$, there exists $Z \in Q_X$ such that $x, x' \in Z$. By the minimality of X, we have $Z \notin S(P(X))$. Consequently, $Z \in Q_X \setminus S(P(X))$ and hence $Z \in \mathcal{U} \subseteq \widetilde{\mathcal{U}}$. □

As a consequence of Theorem 2.4 and Lemma 2.10, we obtain the following.

Corollary 2.15. *Let $k \geq 2$ be an integer, and let P and P' be two posets on the same vertex set V, with $|V| \geq k+3$. If P and P' are $\{-k\}$-hypomorphic, then P and P' are hereditarily hypomorphic.*

Proof. If $k \geq 3$ and P and P' are $\{-k\}$-hypomorphic, then by Lemma 2.10, P and P' are $\{2,3\}$-hypomorphic, and thus by Corollary 2.2, P and P' are hereditarily hypomorphic. Now assume that $k = 2$. The posets P and P' are $\{-2\}$-hypomorphic, so by Lemma 2.10, P and P' are $\{2\}$-hypomorphic. Since P and P' are $\{-2\}$-hypomorphic, then for every $x \in V$, $P - x$ and $P' - x$ are $\{-1\}$-hypomorphic, so $P - x$ and $P' - x$ are $\{-1,2\}$-hypomorphic, and thus by Theorem 2.3, $P - x \simeq P' - x$. Consequently, P and P' are $\{-1,2\}$-hypomorphic, and by Theorem 2.4, P and P' are hereditarily hypomorphic. □

Another consequence of Theorem 2.4, is a short proof of the following result obtained by J. X. Rampon and P. Ille [7].

Corollary 2.16 ([7])**.** *Given a poset P with $n \geq 4$ vertices, P and P^* are $\{-1\}$-hypomorphic if and only if P is a lexicographical sum of total orders over an empty order.*

A poset P is *selfdual* if $P \simeq P^*$. Let P be a poset and let k be an integer, with $k < |V(P)|$. The poset P is $\{k\}$-*selfdual* if P and P^* are $\{k\}$-hypomorphic. The poset P is *strongly selfdual* if P is hereditarily hypomorphic to P^*. From Corollaries 2.15 and 2.16, we get the following result.

Corollary 2.17. *Let $k \geq 1$ be an integer, and P a poset with $|V(P)| \geq k+3$. Then P is strongly selfdual if and only if P is $\{-k\}$-selfdual.*

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