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CONFIGURATION GRAPHS OF NEIGHBOURHOOD GEOMETRIES

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Dedicated to the centenary of the birth of Ferenc Kárteszi (1907–1989).

Abstract. Configurations of type $(\kappa^2+1)_\kappa$ give rise to κ–regular simple graphs via *configuration graphs*. On the other hand, *neighbourhood geometries* of C_4–free κ–regular simple graphs on κ^2+1 vertices turn out to be configurations of type $(\kappa^2+1)_\kappa$. We investigate which configurations of type $(\kappa^2+1)_\kappa$ are equal or isomorphic to the neighbourhood geometry of their configuration graph and conversely. We classify all such graphs and configurations for $\kappa = 3$ and for $\kappa = 4$ when the graphs admit a centre of radius 2.

1. Introduction

For notions from graph theory and incidence geometry, we respectively refer to [3] and [6]. We consider undirected connected graphs without loops and multiple edges.

We call an incidence structure (in the sense of [6]) *linear* if two different points are incident with at most one line. A *configuration of type* n_κ is a linear incidence structure consisting of n points and n lines such that each point and line is respectively incident with κ lines and points.

With each configuration $\mathcal{C}$ of type n_κ, we associate its *configuration graph* $\Gamma(\mathcal{C})$ as the result of the following operation Γ: the vertices of $\Gamma(\mathcal{C})$ are the points of $\mathcal{C}$; any two vertices are joined by an edge if they are not incident with one and the same configuration–line (cf. [9]). Let $\delta := n - \kappa^2 + \kappa - 1$ be the *deficiency* of a configuration of type n_κ (cf. [8, 15]). Note that finite projective planes are characterized by deficiency 0 and, in general, δ indicates the number of points not joined with an arbitrary point. Thus the configuration graph $\Gamma(\mathcal{C})$ is a δ–regular graph on n vertices.

Figure 1 illustrates that if Γ is applied to the Desargues configuration (using Cayley's labelling (cf. [5])), the resulting graph turns out to be the Petersen graph (labelled as *Kneser graph* $KG(2,5)$, cf. [7, 9]).

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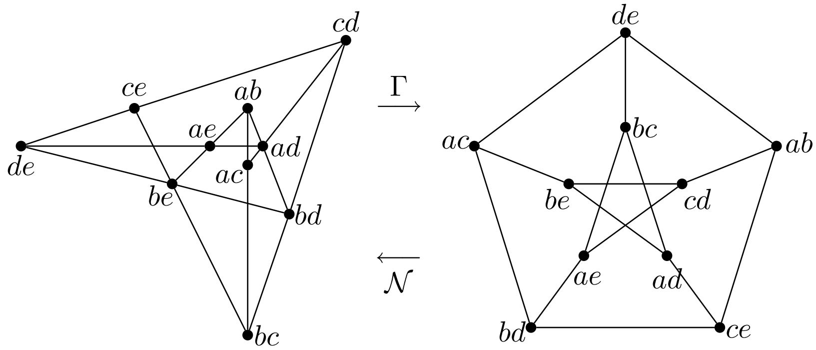


FIGURE 1

Recently, Lefèvre-Percsy, Percsy, and Leemans [14] introduced an inverse operation $\mathcal{N}$ which associates with each graph G its *neighbourhood geometry* $\mathcal{N}(G) = (P, B, |)$: let P and B be two copies of $V(G)$, whose elements are respectively called *points* and *blocks*; a point $x \in P$ is incident with a block $b \in B$ (in symbols $x \mid b$) if, and only if, x and b, seen as vertices in G, are adjacent.

It is easy to check that the neighbourhood geometry of the Petersen graph is the Desargues configuration. Thus, the Desargues configuration and the Petersen graph are invariant under the compositions $\mathcal{N} \circ \Gamma$ and $\Gamma \circ \mathcal{N}$, respectively. (See Figure 1.)

In this paper, we investigate compositions of the operations $\mathcal{N}$ and Γ in a general framework to determine configurations and graphs fixed or isomorphic to the image under these compositions.

Note that a neighboorhood geometry need not be linear, i.e., the forbidden substructure of a linear incidence structure, namely a di-gon $(\{p_1, p_2\}, \{l_1, l_2\}, \{(p_i, l_j) \mid i, j = 1, 2\})$ might show up. This occurs if and only if the graph has a 4–cycle p_1, l_1, p_2, l_2. Therefore, given a κ–regular graph G on n vertices without 4–cycles (i.e., C_4*–free*), its neighbourhood geometry $\mathcal{N}(G)$ is a configuration of type n_κ. On the other hand, given a configuration $\mathcal{C}$ of type $(\kappa^2+1)_\kappa$ its configuration graph $\Gamma(\mathcal{C})$ is κ–regular on κ^2+1 vertices, but not necessarily C_4–free. Hence, a C_4–free κ–regular graph G on κ^2+1 vertices is called $(\Gamma \circ \mathcal{N})$*–admissible*, while a $(\kappa^2+1)_\kappa$ configuration $\mathcal{C}$ is said to be $(\mathcal{N} \circ \Gamma)$*–admissible* if its configuration graph $\Gamma(\mathcal{C})$ is C_4–free.

If both admissibility conditions hold, we can reiterate the compositions of the operations $\mathcal{N}$ and Γ and ask the following questions.

Questions 1.

(1) *Which configurations of type $(\kappa^2+1)_\kappa$ are equal or isomorphic to the neighbourhood geometries of their configuration graphs?*

(2) *Which C_4–free κ–regular graphs on $\kappa^2 + 1$ vertices are equal or isomorphic to the configuration graphs of their neighbourhood geometries?*

We give an answer to these questions for $k = 3$ in Proposition 4.2 and a partial answer for $k = 4$ in Theorem 4.6. To this purpose, we first consider the following weaker questions.

Questions 2.

(1) *Which configurations of type $(\kappa^2 + 1)_\kappa$ are invariant under a power of $\mathcal{N} \circ \Gamma$?*
(2) *Which C_4–free κ–regular graphs on κ^2+1 vertices are invariant under a power of $\Gamma \circ \mathcal{N}$?*

Note that if G is a C_4–free κ–regular graph on $\kappa^2 + 1$ vertices and $(\Gamma \circ \mathcal{N})(G)$ is isomorphic to G, then there exists $n \in \mathbb{N}$ such that $(\Gamma \circ \mathcal{N})^n(G) = G$, but the converse does not necessarily hold (similarly for configurations). Hence, Questions 1 and 2 are not equivalent, and the graphs and configurations answering Question 1 are contained within those answering Question 2. However, Questions 1(1) and 1(2) are equivalent, as well as are Questions 2(1) and 2(2).

In Section 2 we investigate C_4–free κ–regular graphs on $\kappa^2 + 1$ vertices, called $DL\kappa$–graphs for short, and we obtain some restrictions on their diameter and girth. In Section 3 we define an operator Ω which acts on $DL\kappa$-graphs and describes the operation $\Gamma \circ \mathcal{N}$ (Remark 3.6). In this way Question 2 can be reformulated as follows:

Question 3. *Which $DL\kappa$–graphs are invariant under the action of a power of Ω?*

2. $DL\kappa$–Graphs

Let A be a $(0, 1)$–matrix. The matrix A is *doubly stochastic* (of *order* n and *rank* k) if it has order n and there are κ entries 1 in each row and column. The matrix A is *linear* if it does not contain any submatrix of order 2 all of whose entries are 1. In the former case, we say that A fulfills condition (D), in the latter case that it fulfils condition (L). Furthermore, A satisfies conditions (S) and (Z) if it is respectively symmetric and all entries in the main diagonal are zero.

Let $\mathcal{C}$ be a configuration. Fix a labelling for the points and lines of $\mathcal{C}$ and consider the *incidence matrix* $H_\mathcal{C}$ of $\mathcal{C}$ (cf. [6, pp. 17–20]): there is an entry 1 and 0 in position (i, j) of $H_\mathcal{C}$ if and only if the point p_i and the line l_j are incident and non–incident, respectively. The matrix $H_\mathcal{C}$ is a $(0, 1)$–matrix fulfilling conditions (D) and (L).

Remark 2.1. Any $(0, 1)$–matrix fulfilling conditions (D) and (L) gives rise to a configuration of type n_κ. Obviously, the adjacency matrix A_G of a κ–regular graph G satisfies properties (D), (S), and (Z).

Lemma 2.2. *A graph G is C_4–free if and only if its adjacency matrix A_G satisfies condition (L).*

Proof. Let $v_iv_jv_lv_mv_i$ be a 4–cycle in G. This happens if and only if in A_G we find entries 1 in positions $(i,j),(j,l),(l,m)$, and (m,i). By symmetry, this is equivalent to claiming that there are entries 1 in positions $(i,j),(l,j),(l,m)$, and (i,m). This, in turn, happens if and only if the 2×2 submatrix made up by the i^{th} and l^{th} rows and the j^{th} and m^{th} columns has all entries 1. □

Remark 2.3. The adjacency matrix A_G of a C_4–free κ–regular graph on κ^2+1 vertices G coincides with the incidence matrix $H_{\mathcal{N}(G)}$ of the configuration $\mathcal{N}(G)$, whereas the incidence matrix $H(\mathcal{C})$ of a configuration $\mathcal{C}$ need not coincide with the adjacency matrix $A_{\Gamma(\mathcal{C})}$ of its configuration graph $\Gamma(\mathcal{C})$.

Lemma 2.4. *Let $\mathcal{C}$ be a configuration of type n_κ which can be represented as the neighbourhood geometry of some κ–regular graph G. Then $\mathcal{C}$ admits a symmetric incidence matrix whose diagonal entries are zero, i.e., it fulfils (S) and (Z).*

Proof. This follows from Remarks 2.1 and 2.3. □

Remark 2.5. $(\Gamma\circ\mathcal{N})$–admissible graphs and $(\mathcal{N}\circ\Gamma)$–admissible configurations are characterized by $(0,1)$–matrices of order κ^2+1 satisfying conditions (D), (L), (S), and (Z), called $DLSZ\kappa$–matrices, for short.

A *$DL\kappa$–graph* is a C_4–free κ–regular graph on κ^2+1 vertices, i.e., a $(\Gamma\circ\mathcal{N})$–admissible graph.

Proposition 2.6. *A $DL\kappa$–graph has girth 3 or 5.*

Proof. Let g denote the girth of G. A lower bound for the order of any κ–regular graph of girth g is given by the Moore bound (cf. [11, p.184]):

$$f_0(\kappa,g) = \begin{cases} 1 + \dfrac{\kappa((\kappa-1)^{\frac{g-1}{2}}-1)}{\kappa-2} & \text{if } g \text{ is odd,} \\[2ex] 2\dfrac{(\kappa-1)^{\frac{g}{2}}-1}{\kappa-2} & \text{if } g \text{ is even.} \end{cases}$$

In our case, we have

$$f_0(\kappa,g) \le \kappa^2+1. \tag{2.1}$$

Case 1. g even.

It is easy to check that if $k \ge 3$ then $g \le 4$ in (2.1), a contradiction since G is C_4–free. Obviously, the case $g=2$ is ruled out since our graphs G are simple without loops. Hence, we do not have any solution to (2.1) in the even girth case.

Case 2. g odd.

It is easy to check that for $k \ge 3$ we have $g \le 5$ in (2.1), thus $g \in \{3,5\}$. Moreover, equality holds in (2.1) if and only if $g=5$.

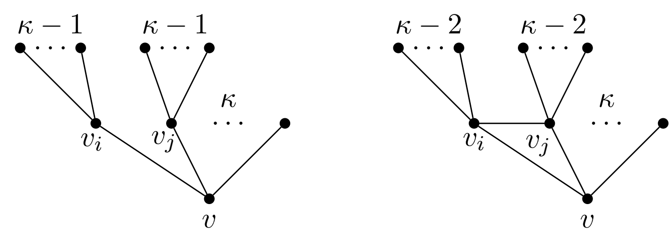
□

Ferenc Kárteszi [13] was the first one to consider k–regular graphs of given girth and he proved the lower bound for the case $g = 6$. We denote by $diam(G)$ the diameter of a graph G and by $d_G(u, v)$ the distance between any two vertices $u, v \in V(G)$.

Proposition 2.7. *Let G be a $DL\kappa$–graph. Then $diam(G) \leq 3$. In particular, $diam(G) = 2$ if and only if the girth of G is 5.*

Proof. Since G is κ–regular, for each vertex $v \in V(G)$ we encounter precisely κ vertices at distance 1, say $v_1, \ldots, v_\kappa$, and at most $\kappa(\kappa - 1)$ vertices at distance 2.

First suppose that $diam(G) \geq 4$, thus there exist vertices $v, w \in V(G)$ with $d_G(v, w) \geq 4$. Denote by $v_1, \ldots, v_\kappa$ and $w_1, \ldots, w_\kappa$ the vertices adjacent with v and w, respectively. Note that $v_i \neq w_j$, for $i, j = 1, \ldots, k$. There might exist possible edges of type v_iv_j and w_lw_m, but no more than one for each vertex v_i and w_l since G is C_4–free. Thus according to whether κ is even or odd there are at least $\kappa(\kappa-2)$ and $(\kappa-1)^2$ vertices at distance 2 from v and w. Hence in both cases the total number of pairwise distinct vertices encountered thus far equals $\kappa^2 + 2$ and $\kappa^2 + 3$, respectively, a contradiction since $|V(G)| = \kappa^2 + 1$. Therefore $diam(G) \leq 3$.



Now suppose $diam(G) = 2$. Clearly, there are $\kappa(\kappa - 1)$ pairwise distinct vertices at distance 2 only if no two out of $v_1, \ldots, v_\kappa$ are adjacent, since each possible edge v_iv_j would reduce by 2 the number of vertices at distance 2. Hence, any vertex $v \in V(G)$ does not lie in a 3–cycle of G, i.e., G does not contain a 3–cycle. Thus, by Proposition 2.6, the girth of G is 5.

Vice versa, if the girth of G is 5, for each vertex v we encounter κ and $\kappa(\kappa - 1)$ distinct vertices at distance 1 and 2 from v, respectively. Hence $diam(G) = 2$ since these vertices, together with v, make up all of $V(G)$. $\square$

Any vertex c of a graph G is called a *centre of G with radius* 2 if $d_G(c, v) \leq 2$, for each $v \in V(G)$. In general, a graph G with $diam(G) = 3$ need not have a centre with radius 2. It is easy to check that every $DL\kappa$–graph G admits a centre with radius 2 if $2 \leq \kappa \leq 3$.

Lemma 2.8. *Let G be a $DL\kappa$–graph not admitting a centre with radius 2. Then every vertex of G lies in a 3–cycle. Moreover, there exists at least one vertex belonging to at least two distinct 3–cycles.*

Proof. By hypothesis, for every vertex v of G there exists at least one vertex w at distance 3 and $\kappa \geq 4$. Denote the vertices at distance 1 from v by $v_1, \dots, v_\kappa$. If v does not lie in a 3–cycle, for $i = 1, \dots, \kappa$, we encounter other $\kappa - 1$ vertices $v_{ij}, j = 1, \dots, \kappa - 1$, at distance 1 from each v_i. Since G is C_4–free, the vertices v_{ij} are pairwise distinct. Hence there are no vertices at distance 3 from v since $V(G) = \{v, v_i, v_{ij} \mid i = 1, \dots, \kappa; j = 1, \dots, \kappa - 1\}$, a contradiction.

It is immediate to check that for every integer κ one has $\kappa^2 + 1 \not\equiv 0 \pmod 3$. Hence $V(G)$ cannot be partitioned into disjoint 3–cycles. □

3. The Operator Ω

In this section, we determine the effect of the composition $\Gamma \circ \mathcal{N}$ on $DL\kappa$–graphs. Recall that by Remark 2.3, for a $DL\kappa$–graph G and its neighbourhood geometry $\mathcal{N}(G)$, one has $A_G = H_{\mathcal{N}(G)}$, whereas the incidence matrix $H(\mathcal{C})$ of a configuration $\mathcal{C}$ need not coincide with the adjacency matrix $A_{\Gamma(\mathcal{C})}$ of its configuration graph $\Gamma(\mathcal{C})$.

Lemma 3.1. *Let $\mathcal{C}$ be a configuration of type n_κ with incidence matrix $H_\mathcal{C}$. Then the adjacency matrix $A_{\Gamma(\mathcal{C})}$ of the configuration graph $\Gamma(\mathcal{C})$ is given by*

$$A_{\Gamma(\mathcal{C})} = (\kappa - 1)I_n + J_n - H_\mathcal{C}(H_\mathcal{C})^{\mathrm{T}},$$

where I_n is the identity matrix of order n and J_n is the square matrix of order n all of whose entries are 1.

Proof. Let $M := (m_{i,j}) := H_\mathcal{C}(H_\mathcal{C})^{\mathrm{T}}$. An arbitrary entry $m_{i,j}$ of M is the result of the usual dot product (over $\mathbb{R}$) of the i^{th} row and the j^{th} row of $H_\mathcal{C}$. Since the rows represent the i^{th} and j^{th} points of $\mathcal{C}$, say p_i and p_j, we have

$$m_{i,j} = \begin{cases} \kappa & \text{if } i = j; \\ 1 & \text{if } i \neq j \text{ and there is a line in } \mathcal{C} \text{ joining } p_i, p_j; \\ 0 & \text{if } i \neq j \text{ and } p_i, p_j \text{ are not joined by any line of } \mathcal{C}. \end{cases}$$

On the other hand, the adjacency matrix $A_{\Gamma(C)} := (a_{i,j})$ of the configuration graph $\Gamma(\mathcal{C})$ has entries:

$$a_{i,j} = \begin{cases} 1 & \text{if the points } p_i, p_j \text{ are not joined by any line of } \mathcal{C}; \\ 0 & \text{otherwise.} \end{cases}$$

This implies $A_{\Gamma(C)} = (\kappa - 1)I_n + J_n - M$. □

Let

$$\Theta : DLSZ\kappa\text{–matrices} \longrightarrow (0,1)\text{–matrices}$$

be the matrix operator given by $\Theta(H) = (\kappa - 1)I_{\kappa^2+1} + J_{\kappa^2+1} - H^2$.

Remark 3.2. (1) Lemma 3.1 implies that Θ describes the action of Γ in terms of incidence matrices of configurations.

(2) Question 1 is equivalent to looking for solutions of the matrix equation

$$H = Q\Theta(H)Q^{-1} \tag{3.1}$$

within the class of $DLSZ\kappa$–matrices, for some permutation matrix Q. We have studied such a problem in [1].

Recall that the *Hamming distance* Δ of two $(0,1)$–vectors is the number of positions where the two vectors have different entries. Note that, in a doubly stochastic $(0,1)$–matrix of rank κ satisfying condition (L), any two of its rows have Hamming distance either 2κ or $2\kappa-2$. The following proposition plays a special role in solving the case $\kappa = 4$ (cf. Section 4.2).

Proposition 3.3. *(Condition (P)) Let H be a doubly stochastic $(0,1)$–matrix of order κ^2+1 and rank κ that satisfies condition (L) and is a solution of (3.1). Then H does not contain any four rows, say $H^{(i)}, H^{(j)}, H^{(l)}$, and $H^{(m)}$, such that*

$$\Delta(H^{(i)}, H^{(l)}) = \Delta(H^{(i)}, H^{(m)}) = \Delta(H^{(j)}, H^{(l)}) = \Delta(H^{(j)}, H^{(m)}) = 2\kappa.$$

Proof. Let $\nu := \kappa^2+1$ and suppose there exist four rows $H^{(i)}, H^{(j)}, H^{(l)}$, and $H^{(m)}$, such that

$$\Delta(H^{(i)}, H^{(l)}) = \Delta(H^{(i)}, H^{(m)}) = \Delta(H^{(j)}, H^{(l)}) = \Delta(H^{(j)}, H^{(m)}) = 2\kappa.$$

Then the entries of the matrix $(\kappa-1)I_\nu + J_\nu - H^2$ in positions (i,l), (i,m), (j,l), and (j,m) are all 1, i.e., the matrix $\Theta(H) = Q^{-1}HQ$ contains a submatrix of order 2 all of whose entries are 1 while H does not, since H is linear, a contradiction. □

Let G be $DL\kappa$–graph. Embed G into the complete graph K_ν, where $\nu := \kappa^2+1$. Then $\kappa\nu/2$ out of $|E(K_\nu)| = \nu(\nu-1)/2$ edges are also edges of G and will be called *G–edges*. Every G–edge either lies in a 3-cycle of G or does not and we will respectively refer to them as a *t–edge* and an *nt–edge*. By Proposition 2.7, the remaining edges of K_ν represent all the pairs of distinct vertices of G having distance either 2 or 3 in G. We respectively call them *d2–edges* and *d3–edges*. Obviously, the edge set $E(K_\nu)$ partitions into the four subsets E_t, E_{nt}, E_{d2}, and E_{d3} of all t–, nt–, $d2$–, and $d3$–edges, respectively. We denote by $G^\star$ the graph K_ν with this partition of $E(K_\nu)$.

The following is an immediate consequence of the definitions of Γ and $\mathcal{N}$.

Lemma 3.4. *Let G be a $DL\kappa$–graph and $u,\ v \in V(G) = V(\Gamma(\mathcal{N}(G)))$. Then the following are equivalent:*

(1) *$uv \in E(\Gamma(\mathcal{N}(G)))$;*
(2) *u, v are not collinear in $\mathcal{N}(G)$;*
(3) *u, v do not have a common neighbour in G.*

Corollary 3.5. *Let G be a $DL\kappa$–graph and $u,\ v \in V(G)$. Then the following hold:*

(1) *If $uv \in E(G)$ then uv is a t-edge in $G^\star$ if and only if $uv \notin E(\Gamma(\mathcal{N}(G)))$.*

(2) *If $uv \notin E(G)$ then uv is a $d3$-edge in $G^\star$ if and only if $uv \in E(\Gamma(\mathcal{N}(G)))$.*

Let G be a $DL\kappa$–graph. We define $\widehat{G} := G^\star[E_{nt} \cup E_{d3}]$ to be the edge–induced subgraph of $G^\star$. Note that, $\widehat{G}$ need not be connected or C_4–free. We define a $D\kappa$–graph to be a (not necessarily C_4–free) κ–regular graph on $\kappa^2 + 1$ vertices. We introduce the operator

$$\Omega : DL\kappa\text{–graphs} \longrightarrow D\kappa\text{–graphs}$$

given by $\Omega(G) = \widehat{G}$.

Remark 3.6. (1) Corollary 3.5 implies that the operator Ω describes $\Gamma \circ \mathcal{N}$, and $\Omega(G)$ is κ–regular.
(2) A necessary condition to re–iterate Ω is that $\widehat{G}$ is C_4–free.

Lemma 3.7. *Let G be a $DL\kappa$–graph. Then $|E_t| = |E_{d3}|$ in $G^\star$.* □

Moreover the operator Ω describes the operator Θ:

Lemma 3.8. *Let G be a $DL\kappa$–graph with adjacency matrix A. Then $\Theta(A)$ is an adjacency matrix for $\widehat{G}$.*

Proof. This follows from Remarks 2.3, 3.2(1) and 3.6(1). □

Corollary 3.9. *Let G be a $DL\kappa$–graph. Then $\Omega(G) = G$ if and only if $diam(G) = 2$ (or equivalently the girth of G is 5).*

Proof. Suppose that $\Omega(G) = G$, i.e., $G^\star$ does not contain any $d3$–edge. Equivalently, $G^\star$ does not contain any t–edge by Lemma 3.7. Thus, we have $\Omega(G) = G$ if and only if all edges of G are nt–edges, i.e., G has no 3–cycle. Hence, Proposition 2.6 implies that the girth of G is 5, i.e., $diam(G) = 2$, by Proposition 2.7. □

In 1960, Hoffman and Singleton [10] classified all κ–regular graphs G on $\kappa^2 + 1$ vertices having girth 5. There are at most four of them, namely:

(1) $\kappa = 2$ and G is the 5–cycle;
(2) $\kappa = 3$ and G is the Petersen graph;
(3) $\kappa = 7$ and G is Hoffman–Singleton's $(5, 7)$–cage;
(4) $\kappa = 57$ (no graph is known).

Hoffman–Singleton's classification reply to Question 3 for Ω–invariant $DL\kappa$–graphs for $\kappa \neq 57$. Note that $\Omega(G) = G$ would eventually hold for a 57–regular graph of girth five on 3250 vertices, if it existed. In terms of configurations, the following gives an answer to Question 1(1) in the equality case.

Theorem 3.10. *Let $\mathcal{C}$ be a configuration of type $(\kappa^2 + 1)_\kappa$ and assume that $\mathcal{C}$ is the neighbourhood geometry of its configuration graph. If $\kappa \neq 57$, one has one of the following cases:*

(1) *$\kappa = 2$ and $\mathcal{C}$ is the pentagon;*
(2) *$\kappa = 3$ and $\mathcal{C}$ is the Desargues configuration;*

(3) *$\kappa = 7$ and $\mathcal{C}$ is the neighbourhood geometry of the Hoffman-Singleton graph.*

Proof. Hoffman–Singleton's classification and Corollary 3.9 imply the result. □

Note that a necessary condition to re-iterate Ω is that $\widehat{G}$ is C_4–free.

Lemma 3.11. *Let G be a $DL\kappa$–graph such that $diam(G) = 3$ and $\Omega^n(G) = G$, for some positive integer $n > 1$. Then any path of length 3 consisting only of nt–edges lies in a 5–cycle of G.*

Proof. Let $P_3 := v_1v_2v_3v_4$ be a path of length 3 in G such that $E(P_3) \subseteq E_{nt}$. Clearly, v_1 and v_4 are not adjacent, otherwise G would not be C_4–free. Suppose that v_1 and v_4 are at distance 3 in G, then $v_1v_4 \in E_{d3}$ in $G^\star$, i.e., $\widehat{G}$ is not C_4–free, a contradiction. Thus $d(v_1, v_4) = 2$. By hypothesis, v_1v_3, $v_2v_4 \in E(G^\star) - E(G)$. Hence, there exists a vertex $v \in V(G) - V(P_3)$ such that $P_3 \cup \{v_1v, vv_4\}$ is the desired 5–cycle containing P_3 in G. □

4. Ω^n–Invariant $DL\kappa$–Graphs

In this Section we give a complete answer to Questions 1, 2 and 3 for $\kappa = 3$ and $\kappa = 4$.

A *quadrangle* Q in a graph G is a 4–cycle such that its induced subgraph $G[Q] = Q$. Quadrangle–free graphs play an important role in the theory of distance–regular graphs (cf. [4, Section 1.16]). Any graph G is quadrangle–free if and only if for any two vertices $v_1, v_2 \in V(G)$ such that $d(v_1, v_2) = 2$, the induced subgraph $G[N_G(v_1) \cap N_G(v_2)]$ is a clique. For a C_4–free graph G, this sharpens to the property that $N_G(v_1)$ and $N_G(v_2)$ intersect in just one vertex, i.e., in a clique of size 1.

Recall that a *Terwilliger graph* is a non–complete graph G such that, for any two vertices $v_1, v_2 \in V(G)$ with $d(v_1, v_2) = 2$, the induced subgraph $G[N_G(v_1) \cap N_G(v_2)]$ is a clique of size μ, for some fixed $\mu \geq 0$ (cf. [4, p. 34]).

Thus the class of $(\Gamma \circ \mathcal{N})$–admissible κ–regular graphs coincides with the class of κ–regular Terwilliger graphs for $\mu = 1$.

4.1. $DL3$–graphs and configurations of type 10_3.

Lemma 4.1. *Let G be a $DL3$–graph. Then G is isomorphic to one of the following Terwilliger graphs with $\mu = 1$:*

(1) *the Petersen graph P;*
(2) *the graph T_1 (cf. Figure 2);*
(3) *the graph T_2 (cf. Figure 2).*

Proof. By Proposition 2.6, the graph G has either girth 5 or girth 3. In the former case, by Hoffman–Singleton's classification G is isomorphic to the Petersen graph P.

In the latter case, any two 3–cycles in G have disjoint vertices and edges since G is cubic and C_4–free. But the 3–cycles cannot span $V(G)$ since

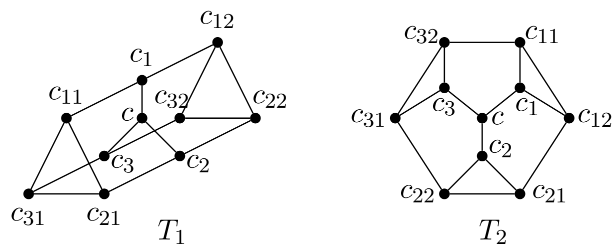


FIGURE 2

$10 \equiv 1 \pmod 3$. Thus there exists a vertex $c \in V(G)$ such that c is a centre of G with radius 2. We denote by $N(c) := \{c_1, c_2, c_3\}$ the neighbours of c and by $N(c_i) := \{c, c_{i1}, c_{i2}\}$ the neighbours of c_i, for $i = 1, 2, 3$.

The subgraph $S := G[c_{ij}]$ induced by the vertices c_{ij} is 2–regular, hence either S is the disjoint union of two 3–cycles or S is a 6–cycle. It is easy to check that, in the former case, G is isomorphic to the graph T_1 with

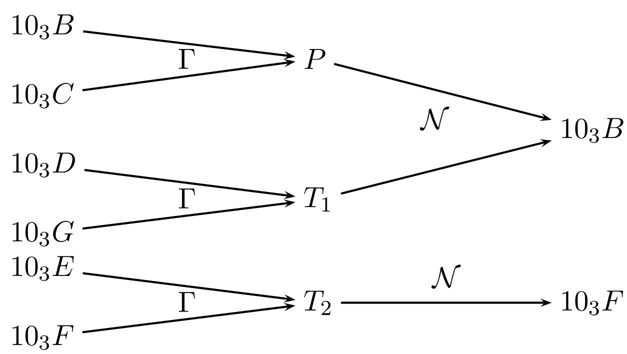
$$E(S) = \{c_{1j}c_{2j}, c_{1j}c_{3j}, c_{2j}c_{3j} \mid j = 1, 2\}.$$

In the latter case, it follows that c_i, c_{i1}, c_{i2} make up a 3–cycle, for each $i = 1, 2, 3$, and G is isomorphic to T_2. □

Kantor [12] denoted the ten configurations of type 10_3 by 10_3A, …, 10_3I, 10_3K. The Desargues configuration corresponds to 10_3B. It is part of mathematical folklore that these ten configurations yield seven pairwise non–isomorphic configuration graphs, namely P, T_1, and T_2, as well as four graphs containing 4–cycles.

The following proposition gives an answer to Question 2(1) in terms of configurations of type 10_3 and to Questions 2(2) and 3 in terms of $DL3$–graphs. Furthermore, the last statement gives an answer to Question 1(1) for $\kappa = 3$. The proof is an easy exercise.

Proposition 4.2. *The Γ–images of the configurations 10_3B, …, 10_3G and the $\mathcal{N}$–images of the graphs P, T_1, T_2 give rise to the following diagram:*



In particular, $\Omega(P) = P$ and $\Omega^3(T_2) = T_2$, while the configurations 10_3B and 10_3F are respectively $(\mathcal{N} \circ \Gamma)$– and $(\mathcal{N} \circ \Gamma)^3$–invariant. Moreover, the *neighbourhood geometry of the configuration graph of* 10_3F *is isomorphic to* 10_3F *and the configuration graph of the neighbourhood geometry of* T_2 *is isomorphic to* T_2*.* □

4.2. $DL4$**–graphs and configurations** 17_4**.** Betten and Betten [2] point out that there exist 1972 pairwise non–isomorphic configurations of type 17_4. They also give a list of all 26 instances whose automorphism groups have orders at least 5. It is an easy exercise to verify that, out of these 26, only the configurations 1917, 1918, 1964, and 1971 in Bettens' list satisfy Condition (P) (cf. Proposition 3.3).

In this subsection we prove that configuration 1971 gives a positive answer to Questions 1(1) and 2(1), and that its configuration graph partially answers Questions 1(2), 2(2) and 3 for $\kappa = 4$ (cf. Theorem 4.6).

Let G be a $DL4$–graph which does not have a centre with radius 2. Then Lemma 2.8 implies that every vertex $v \in V(G)$ lies in a 3–cycle and there exists at least one vertex belonging to exactly two distinct 3–cycles. The calculations performed so far have given no graph G fulfilling these conditions. Hence we propose the following:

Conjecture. *Let* G *be a* $DL4$*–graph such that* $\Omega^n(G) = G$*, for an integer* $n > 1$*. Then* G *admits a centre with radius* 2*.*

From now on, we denote by G_{17} a $DL4$–graph having a centre z with radius 2. Let G_z be the cubic C_4–free subgraph of G_{17} of order 12 induced by the vertices at distance 2 from z. We say that G_z is the *periphery* of z in G_{17}.

Let G be cubic graph, let $v \in V(G)$ and $N(v) = \{w_1, w_2, w_3\}$. We define the *blow up* of v in G the operation that deletes v and adds the 3–cycle $v_1v_2v_3v_1$ and the edges v_iw_i, $i = 1, 2, 3$. This operation transforms G into a new cubic graph G' of order $|V(G')| = |V(G)| + 2$. The inverse of such an operation is the *contraction* of the 3–cycle $v_1v_2v_3v_1$ to the vertex $v \in V(G)$.

The following table lists four relevant examples of cubic graphs obtained by blowing up one vertex of the Petersen graph P and the Terwilliger graphs T_1 and T_2.

Graph	Blow up
P'	any vertex of the Petersen graph P
T_1'	the centre c of the graph T_1
T_2'	the centre c of the graph T_2
T_1''	a neighbour c_i of the centre c of the graph T_1, $i = 1, 2, 3$.

It is well known that there exist 85 cubic graphs of order 12 none of which has girth ≥ 6, two have girth 5, say H_1 and H_2, twenty have girth 4 and the remaining 63 have girth 3 [16].

Lemma 4.3. *There exist precisely eight* C_4*–free cubic graphs of order* 12*, namely the two graphs* H_1 *and* H_2 *of girth* 5*, the four graphs* P'*,* T_1'*,* T_2'*,* T_1'' *and two more graphs* Q' *and* Q'' *obtained from the cube.*

Proof. Let G be a C_4–free cubic graph of order 12. Clearly, the cases of girth 4 and girth ≥ 6 are ruled out by the hypotheses (cf. [16]). If G has girth 5, we have only H_1 and H_2. Thus, suppose that G has girth 3.

The contraction of a 3–cycle in G gives rise to a cubic graph of order 10 and it is well known that there are precisely 19 graphs of this kind (cf. [16]). Note that blowing up of a vertex belonging to a 4–cycle removes such a cycle, while blowing up a 3–cycle produces a 4–cycle. Therefore, a cubic graph of order 10 containing two disjoint 4–cycles cannot be transformed into a C_4–free cubic graph of order 12 via a blow up. It is an easy exercise to check that there are 13 out of the 19 cubic graphs of order 10 which contain two disjoint 4–cycles. The six remaining cubic graphs are the Petersen graph P, the Terwilliger graphs T_1, T_2, two cubic graphs obtained by blowing up a vertex of the cube graph Q_3 and the twisted cube Q'_3 (i.e., the cubic graph obtained from Q_3 by twisting the edges of one of its 4–cycles), and an additional graph referred to as H_3 (i.e., a connected cubic graph with exactly two 3–cycles and one 4–cycle which are mutually disjoint) [16].

Since G is C_4–free, the only possibilities of blowing up P, T_1 and T_2 are those described in the table above, and we obtain P', T'_1, T'_2 and T''_1. Similarly, the only possibilities of blowing up Q_3 and Q'_3 are using a vertex opposite to the 3–cycle in Q_3 and Q'_3, respectively. In this way we get two more graphs Q' and Q''. Finally, the only possibility to blow up H_3 is via a vertex of its 4–cycle, the resulting graph is isomorphic to T''_1. □

Next we determine which of the eight graphs listed in Lemma 4.3 can appear as the periphery of a $DL4$–graph.

Lemma 4.4. *Let G_{17} a $DL4$–graph having a centre z with radius 2. Then H_1, P', T'_1, T''_1 and T'_2 are the only possible peripheries G_z in G_{17}.*

Proof. Let z be a centre with radius 2 in G_{17} and let $N(z) = \{z_1, z_2, z_3, z_4\}$ its neighbours in G_{17}. Let G_z be the periphery of z in G_{17}. Note that its 12 vertices are neighbours of $N(z)$ in G_{17}. Since G_{17} is 4–regular and C_4–free, $V(G_z)$ can be partitioned into exactly four triples which are the sets $N(z_i)\backslash\{z\}$, for $i = 1, \ldots, 4$. The vertices in each triple are either at distance 3 or adjacent, but the edge between them does not belong to a 3–cycle, i.e., it is not a t–edge. It is a lengthy but easy exercise to check that the only C_4–free cubic graphs of order 12 admitting such a partition are H_1, P', T'_1, T''_1 and T'_2. □

Lemma 4.4 immediately implies that the five peripheries give rise to the following $DL4$–graphs having centre z with radius 2:

Periphery	$DL4$-graphs
H_1	$G_{17}(H_1)$
P'	$G_{17}(P')$
T'_1	$G_{17}(T'_1)$
T''_1	$G_{17}(T''_1)$
T'_2	$G^{(1)}_{17}(T'_2)$ and $G^{(2)}_{17}(T'_2)$

We have computed that $G_{17}(H_1)$ and $G_{17}(P')$ are, respectively, isomorphic to $G_{17}(T_1')$ and $G_{17}^{(1)}(T_2')$.

Corollary 4.5. *Let G_{17} be a $DL4$–graph having centre z with radius 2. Then G_{17} is isomorphic to one of the following:*

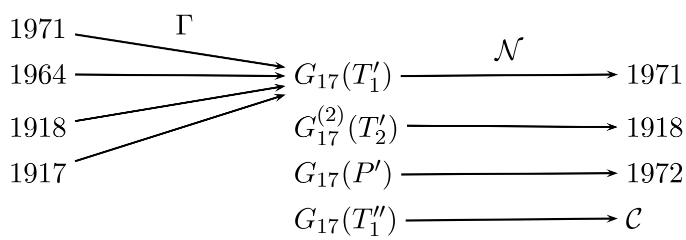
(1) $G_{17}(T_1')$;
(2) $G_{17}(P')$;
(3) $G_{17}(T_1'')$;
(4) $G_{17}^{(2)}(T_2')$.

We conclude the reasoning with an analogue of Proposition 4.2 that gives a partial answer to Question 1, 2 and 3 in the case $\kappa = 4$.

Theorem 4.6. *The Γ–images of the configurations 1917, 1918, 1964 and 1971 and the $\mathcal{N}$–images of the graphs*

$$G_{17}(T_1'),\ G_{17}(P'),\ G_{17}(T_1'') \text{ and } G_{17}^{(2)}(T_2'')$$

give rise to the following diagram:



where $\mathcal{C}$ is a configuration of type 17_4 with $|Aut(\mathcal{C})| = 4$.

In particular, $\Omega^2(G_{17}(T_1')) = G_{17}(T_1')$ and the configuration 1971 is $(\mathcal{N} \circ \Gamma)^2$–invariant. Moreover, the neighbourhood geometry of the configuration graph of configuration 1971 is isomorphic to the configuration 1971 and $(\Gamma \circ \mathcal{N})(G_{17}(T_1')) \cong G_{17}(T_1')$.

Proof. By Corollary 4.5, the only $DL4$–graphs admitting a centre of radius 2 which might be invariant under a power of Ω are $G_{17}(T_1')$, $G_{17}(P')$, $G_{17}(T_1'')$ and $G_{17}^{(2)}(T_2'')$. The graph $\Omega(G_{17}(T_1'))$ is isomorphic to $G_{17}(T_1')$ but their t–edges and d_3–edges are exchanged, whereas $\Omega^2(G_{17}(T_1')) = G_{17}(T_1')$.

Then the statement follows by determining their neighbourhood geometries under $\mathcal{N}$ and the configuration graphs under Γ. Note that the configurations 1972 and $\mathcal{C}$ have configuration graphs which are not C_4–free. □

In this last subsection, we have used the software *Groups and Graphs* by W. Kocay at the University of Manitoba to determine which graphs were isomorphic.

References

1. M. Abreu, M. Funk, D. Labbate, and V. Napolitano, *Adjacency matrices of configuration graphs*, submitted.

2. A. Betten and D. Betten, *Tactical decompositions and some configurations* v_4, J. Geom. **66** (1999), 27–41.
3. J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, North Holland, New York, Amsterdam, London, 1976.
4. A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs*, Springer, Berlin, Heidelberg, New York, 1989.
5. H. S. M. Coxeter, *Self-dual configurations and regular graphs*, Bull. Amer. Math. Soc. **56** (1950), 413–455, Reprinted in *Twelve Geometric Essays*, Southern Illinois University Press, Carbondale, 1968, pp. 106–149.
6. P. Dembowski, *Finite geometries*, Springer, Berlin, Heidelberg, New York, 1968, Reprinted 1997.
7. J. W. Di Paola and H. Gropp, *Hyperbolic graphs from hyperbolic planes*, Congressus Numerantium **68** (1989), 23–44.
8. H. Gropp, *On the existence and non-existence of configurations* n_k, J. Combin. Inform. Systems Sci. **15** (1990), 34–48.
9. ______, *Configurations and graphs*, Discrete Mathematics **111** (1993), 269–276.
10. A. J. Hoffman and R. R. Singleton, *On Moore graphs with diameters* 2 *and* 3, IBM Journal (1960), 497–504.
11. D. A. Holton and J. Sheehan, *The Petersen graph*, Cambridge University Press, Cambridge, 1993.
12. S. Kantor, *Die Configurationen* $(3,3)_{10}$, Sitzungsber. Wiener Akad. **84** (1881), 1291–1314.
13. F. Kárteszi, *Piani finiti ciclici come risoluzioni di un certo problema di minimo*, Boll. Un. Mat. Ital. **3** (1960), no. 15, 522–528, In Italian.
14. C. Lefèvre-Percsy, N. Percsy, and D. Leemans, *New geometries for finite groups and polytopes*, Bull. Belg. Math. Soc. Simon Stevin **7** (2000), 583–610.
15. M. J. Lipman, *The existence of small tactical configurations*, Graphs and Combinatorics (R. A. Bari and F. Harary, eds.), Lecture Notes in Mathematics, no. 406, Springer, Berlin, Heidelberg, New York, 1974, pp. 319–324.
16. G. Royle, *Cubic graphs*, `http://www.csse.uwa.edu.au/$\sim$gordon/remote/cubics/index.html`.

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