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OPTION-CLOSED GAMES

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Abstract. We consider the class of combinatorial games with the property that each player's move eliminates some options but does not add any new options for that player. While the canonical form can be complicated, we show that the reduced canonical form of a position is either a number or a switch. Moreover, for a given position, the difference between the two canonical forms is bounded by $\Downarrow *$ and $\Uparrow *$.

1. Introduction

The game of Roll the Lawn uses a row of bumps (nonnegative integers) and a roller that is either between two bumps or at one end of the row. Left moves the roller to the left flattening each bump by 1 unless the bump has been flattened to 0 in which case nothing happens to that bump. For a move to be legal, at least one bump must be flattened by 1. Right moves the roller to the right, with the same effect and constraint. This is a combinatorial game played with Normal-play rules, i.e. the last player to move wins. (See [9] for play under the misère rules.) This game can be visualized as being played on a graph where the bumps are weights on the edges but the roller is on a vertex. In this paper, we represent a position by a string of nonnegative integers with a roller, $\ominus$, in the string. We also use $\xrightarrow{L}$ to indicate a Left move, and a Right move by $\xrightarrow{R}$. For example, in the following position, if Right moves first, play might proceed:

$$[5, \ominus, 2, 4] \xrightarrow{R} [5, 1, 3, \ominus] \xrightarrow{L} [5, \ominus, 0, 2] \xrightarrow{R} [5, 0, 1, \ominus] \xrightarrow{L} [5, 0, \ominus, 0]$$

and Left has won.

In Roll the Cricket pitch, or Cricket Pitch for short, there is an extra constraint, the roller cannot go over a bump that has already been flattened to 0. For example:

$$[5, \ominus, 1, 2, 4] \xrightarrow{R} [5, 0, 1, \ominus, 4] \xrightarrow{L} [5, 0, \ominus, 0, 4]$$

and the game is over.

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In both games, the set of options available after a player's move is a subset of the options before his move, i.e. what he can do in two consecutive moves he can also do in one. This is also true for HACKENBUSH STRINGS [3], and MAZE [2]. We make this formal.

Definition 1.1. *A game G is called* option-closed *if $G^{LL} \subset G^L$, $G^{RR} \subset G^R$ and all of the options of G are option-closed.*

It is important to note that option-closed is a property of a game that depends on its form. In particular, given two games with the same canonical form one might be option closed but the other not. For example, $\{1 \mid 0\}$ is not option closed but $\{1, 0 \mid 0\}$ is.

Throughout this paper, when we make reference to a game G that is option-closed, we assume that G is of a form that satisfies the definition given above.

Although the canonical form of an option-closed game can be complicated, we will show that they can be expressed in simpler terms. For example, in CRICKET PITCH;

$$G = [1, 1, 3, \ominus, 5, 3, 1] = \{1,\ \{1|0\} \,|\, 0,\ \{1|0\},\ \{1, \{1|0\}|0, \{1|0\}\}\},$$

but this is $\{1|0\} + G'$ where G' is an infinitesimal (see Definition 2.1). Left's winning move is to $[1, \ominus, 0, 2, 5, 3, 1]$ and Right's is to $[1, 1, 3, 4, 2, 0, \ominus]$. We leave it to the reader to show that all other moves lose. The difference $G - \{1 \mid 0\}$ is an infinitesimal, more precisely, $\mathbf{-}_1 < G - \{1 \mid 0\} < \mathbf{+}_1$ where $\mathbf{+}_1 = \{0 \mid \{0 \mid -1\}\}$ and $\mathbf{-}_1 = \{\{1 \mid 0\} \mid 0\}$. The standard unit for measuring infinitesimals is $\uparrow = \{0 \mid *\}$. The value $k \cdot \mathbf{+}_1$ is not larger than $\uparrow$ for any integer k, so even in the infinitesimal world this value is'very small'.

In the next section, we show that the *reduced canonical* form (see Definition 2.4) of any option-closed game is simple and the difference between a game and its reduced canonical form is no larger than $2 \cdot \uparrow + *$. In the last section, after making a few comments about MAZE, we consider ROLL THE LAWN and CRICKET PITCH in detail. The analysis of ROLL THE LAWN is straightforward, but the analysis of CRICKET PITCH is not complete. Although we give the outcome classes for an individual game, the determination of even the reduced canonical form has eluded us.

See any of [2, 3, 5] for any undefined terms and general combinatorial game theory background.

2. Reduced Canonical Forms of Option-closed Games

In many games, a player would like to ignore the infinitesimal values since they only determine the parity of the number of moves once the associated non-infinitesimal value has reached zero. Grossman & Siegel [7] showed that the idea of a simplest game infinitely close to a given game, called the *reduced canonical form*, is well-defined. (Calistrate [4] first introduced the idea but the proof contained a flaw and a different approach was required.) We need to present a brief development of the tools so that we can apply them to option-closed games.

Definition 2.1. *A game G is an* infinitesimal *if, for every positive number x, we have $-x < G < x$. Let* Inf *denote the set of infinitesimals. When $G - H$ is infinitesimal, we say that G and H are* infinitesimally close*, and write $G \equiv_{\text{Inf}} H$. We will sometimes say that H is G-ish (G Infinitesimally SHifted).*

Definition 2.2 ([7])**.** *$G \geq_{Inf} H$ if $G \geq H + \epsilon$ for some infinitesimal ϵ; $G \leq_{Inf} H$ is defined similarly.*

Definition 2.3 ([7])**.** *Let G be any game.*

- *A Left option G^L is* Inf -dominated *if $G^L \leq_{Inf} G^{L'}$ for some other Left option $G^{L'}$.*
- *A Left option G^L is* Inf -reversible *if $G^{LR} \leq_{Inf} G$ for some G^{LR}.*

The definitions for Right options are similar.

For example, let $G = \{1, \{1|0\}|0\}$. Then $\{1|0\}$ is an Inf-dominated Left option of G, since $\{1|0\} \leq 1 + \uparrow$. If this 'dominated' option could be eliminated, the resulting simpler game would be $\{1|0\}$. In the reduced canonical form, the Inf-dominated options are removed.

Definition 2.4 ([7], 4.2)**.** *A game G is said to be in* reduced canonical form *provided that, for every follower H of G, either:*

- *H is a number in simplest form; or*
- *H is not a number or a number plus an infinitesimal, and contains no Inf-dominated or Inf-reversible options.*

Theorem 2.5 ([7], 4.3)**.** *For any game G, there is a game $\overline{G}$ in reduced canonical form with $G \equiv_{\text{Inf}} \overline{G}$.*

Theorem 2.6 ([7], 4.4)**.** *Suppose that G and H are in reduced canonical form. If $G \equiv_{\text{Inf}} H$, then $G = H$.*

This shows that the *reduced canonical form* of a game G is well-defined and unique. It is denoted $\overline{G}$. The next lemma and theorem are needed to prove results about the reduced canonical form of option-closed games.

Lemma 2.7 ([7], 4.6)**.** *If G is not a number and G' is obtained from G by eliminating an Inf-dominated option, then $G' \equiv_{\text{Inf}} G$.*

Theorem 2.8 ([7], 2.10)**.** *If $G = \{G^L|G^R\}$ is not a number and $G' = \{G^{L'}|G^{R'}\}$ is a game with $G^{L'} \equiv_{\text{Inf}} G^L$ and $G^{R'} \equiv_{\text{Inf}} G^R$, then $G' \equiv_{\text{Inf}} G$.*

We are now in a position to consider option-closed games.

Lemma 2.9. *For any numbers a, b with $b \leq a$, then $a \geq_{\text{Inf}} \{a|b\}$.*

Proof. Consider $a - \{a|b\} + n \cdot \uparrow = a + \{-b| - a\} + n \cdot \uparrow$ with $n \geq 3$. By the Number Avoidance Theorem ([2], Theorem 6.17, page 125), if Right can win then he must have a good move in either $\{-b| - a\}$ or in $n \cdot \uparrow$. The former leaves $a - a + n \cdot \uparrow > 0$, while the latter leaves $a + \{-b| - a\} + (n-1) \cdot \uparrow *$, in which case Left responds to $a - b + (n-1) \cdot \uparrow * > 0$. Since Left wins in both cases then $a \geq \{a|b\} + n \cdot \downarrow$, i.e. $a \geq_{\text{Inf}} \{a|b\}$. □

Theorem 2.10. *Let G be a option-closed game. The reduced canonical form of G is either a number or a switch, $\{a|b\}$, where a and b are numbers.*

Proof. We proceed by induction and use Lemma 2.7 and Theorem 2.8.

Suppose G is the simplest option-closed game whose reduced canonical form is not a switch or a number. Thus, the reduced canonical forms of all the Left options of G are all switches or numbers. By Lemma 2.7 and Theorem 2.8, any Left option can be replaced by its reduced canonical form. If the switch $\{a|b\}$ is a Left option then so is a. Now by Lemma 2.9, $a \geq_{\text{Inf}} \{a|b\}$ and Lemma 2.7 and Theorem 2.8 the switch is Inf-dominated and can be removed. □

Definition 2.11. [3, 2] *Denote the* left stop *and* right stop *of a game G by LS(G) and RS(G), respectively. They are defined in a mutually recursive fashion:*

$$\textbf{LS}(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \max(\textbf{RS}(G^L)) & \text{if } G \text{ is not a number;} \end{cases} \tag{2.1}$$

$$\textbf{RS}(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \min(\textbf{LS}(G^R)) & \text{if } G \text{ is not a number.} \end{cases} \tag{2.2}$$

Lemma 2.12. *Let G be game and x a number. If $x > \textbf{LS}(G)$ then $x > G$; if $x < \textbf{RS}(G)$ then $x < G$; and if $\textbf{RS}(G) < x < \textbf{LS}(G)$ then $x \parallel G$.*

Proof. Theorem 6.11 of [2] states that $\mathbf{LS}(G) \geq \mathbf{RS}(G)$ (also see [5], page 99).

We amend the proof to obtain the above result.

If $x > \mathbf{LS}(G)$ then, when playing $x - G$, the Weak Number-Avoidance Theorem asserts that if there is a winning move, it is in G. So, without loss of generality, both players play on G until it reaches a number. When Left moves first, the maximum she can achieve is $\mathbf{LS}(G)$, and when Right moves first, the minimum he can achieve is $\mathbf{RS}(G)$. Neither are good enough for the first player to achieve a win on xG. □

The proofs of the next two Corollaries follow almost directly from Theorem 2.10 and from the definitions of stops and option-closed games.

Corollary 2.13. *Suppose G is option-closed.*

(1) *If $\overline{G} = x$ then $\textbf{LS}(G) = \textbf{RS}(G) = x$. Moreover, if $G \neq x$ then there exists left and right options with $G^L = G^R = x$.*

(2) *If $\overline{G} = \{a|b\}$ then $\boldsymbol{LS}(G) = a$, $\boldsymbol{RS}(G) = b$. Moreover, there exists left and right options with $G^L = a$ and $G^R = b$.*

Proof. We proceed by induction on the birthday of G. The statement is true for $G = 0$.

Suppose it is true for all games born by day n and let G be an option-closed game born on day $n+1$. If G is a number then there is nothing to prove. Let $H \in L(G)$ then one of the following cases holds:

- $H = x_H$ is a number then (trivially) $x_H \in L(G)$;
- $\overline{H} = x_H$ is a number but H is not, then, by induction, $x_H = \mathbf{LS}(H) = \mathbf{RS}(H)$, also $x_H \in L(G)$, and since G is option-closed then $x_H \in L(G)$;
- $\overline{H} = \{x_H|y_H\}$ with x_H, y_H numbers and $x_H > y_H$, also, by induction, $x_H = \mathbf{LS}(H)$, $y_H = \mathbf{RS}(H)$.

Let $A = \{x_H : x_H \in L(G)\}$ and $B = \{y_H : H \in L(G), \overline{H} = \{x_H|y_H\}\}$. Expanding the definition of the left stop, we have $\mathbf{LS}(G) = \max A \cup B$. Now, y_H is only in B if $x_H > y_H$. Since x_H is in A then $\mathbf{LS}(G) = \max A \cup B = \max A$. Therefore the left-stop of G is in $L(G)$.

The arguments for the right-stop are similar. □

Corollary 2.14. *Suppose G is option-closed and suppose $\boldsymbol{LS}(G) = x$. If the left option G^{L_1} has a right option then there is some right option $G^{L_1R_2} = y \le x$ for some number y.*

Note that Corollary 2.13 gives information about the infinitesimals when $\overline{G}$ is a number.

Corollary 2.15. *Let G be a option-closed game. If $\overline{G}$ is a number and $G \neq \overline{G}$ then $G - \overline{G}$ is confused with 0.*

However, these infinitesimals can also be bounded.

Theorem 2.16. *Let G be a option-closed game then $\Downarrow * < G - \overline{G} < \Uparrow *$*

Proof. If $G = \overline{G}$ then the result is true. First, let's assume that $G - \overline{G}$ is positive. We must show that $G - \overline{G} + \Downarrow * < 0$.

We have two cases. First, we assume $\overline{G} = x$ for some number x. Now, we have to show that Right can win $G - x + \Downarrow *$ going first or second. The situation is highlighted by the game tree in Figure 1.

When Right plays first, we know that he can play in G to x since x must be a Right stop of G. Using the fact that G is option-closed tells us he can get to this stop in one move. That leaves us with $\Downarrow * < 0$ so Right wins.

If Left plays first, his only good moves can be in G or $\Downarrow *$. Suppose Left plays in G to G'. If Right has a legal response in G' then, by Corollary 2.14, Right has a move to a number that is at most x. After this pair of moves, it will be Left's turn to move in a position no more than $\Downarrow * < 0$, so she again loses. On the other hand, if Right has no legal move in G', and since $G \neq x$ then, by Corollary 2.13, G' must be a number with value at most x. This

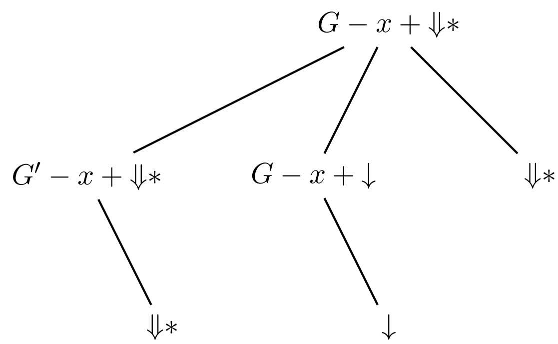


FIGURE 1.

will leave a position with value at most $\Downarrow *$ which Left loses. Therefore, his only possible winning move would have to be $\Downarrow *$ to $\downarrow$. As before, Right wins by replying in G to x. So, Left has no good move and loses regardless of who plays first.

Now, we assume that $\overline{G} = \{a|b\}$ where $a > b$. We need to now show that $G + \{-b|-a\} + \Downarrow * < 0$. If Right plays first, he can move to $b + \{-b|-a\} + \Downarrow *$ by Corollary 2.13. From this position, Left's best move must be in the switch to $b - a + \downarrow$ (by the Number Avoidance theorem and since $a > b$) but this is negative since $b < a$ so Right wins playing first. The moves are highlighted in the game tree in Figure 2.

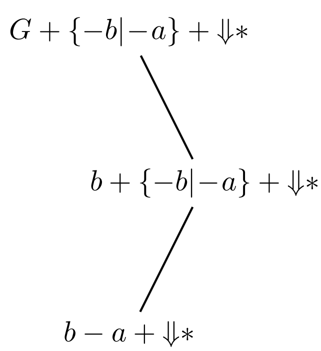


FIGURE 2.

If Left plays first in $G + \{-b \mid -a\} + \Downarrow *$ she has three different components she could play in. These are illustrated in the last game tree.

If she plays to $G - b + \Downarrow *$, Right responds to $\Downarrow *$ and wins. If instead she plays to $G + \{-b| - a\} + \downarrow$, then, by Corollary 2.13, Right can respond to $b + \{-b| - a\} + \downarrow$. From this position, Left's best move must be in the switch which leaves $\downarrow < 0$ and she loses.

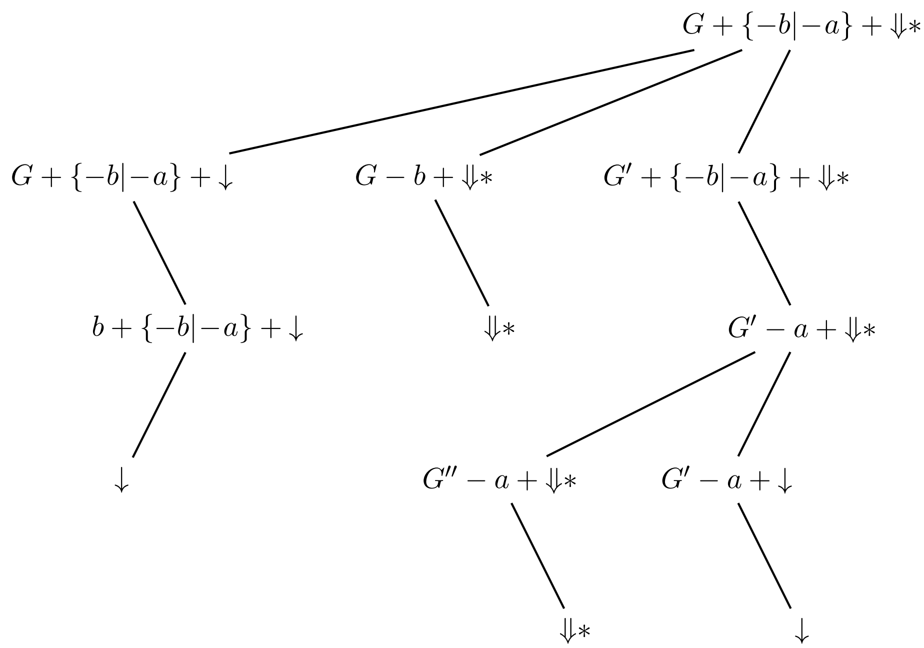


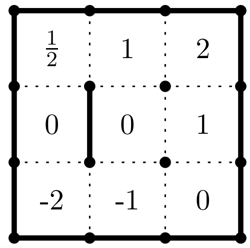
Figure 3.

The final case to consider is when Left plays in G from $G+\{-b|-a\}+\Downarrow *$ (see Figure 3). As before, we will call this position G' and we are left with $G' + \{-b| - a\} + \Downarrow *$. Right has a winning strategy from here by replying to $G' - a + \Downarrow *$ since now Left must either play in $\Downarrow *$ or G' if a move exists for her in that component. In the first case, we are left with $G' - a + \downarrow$. As before, by Corollary 2.14, if Right has a legal move in G' we know it is to a number no larger than a and this is a winning move. If there is no legal move for Right, then we are already in such a position which must have a value of at most $\downarrow$ and, again, it is a win for Right. In the second case, we assume Left has a legal move in G' and plays to G'' leaving us with the position $G''-a+\Downarrow *$. As before, by Corollary 2.14, either Right can move in G'' to a game with value at most $\Downarrow *$, or Right has no move in G'' in which case we are already in a position with value at most $\Downarrow *$. Both cases lead to a Right win.

□

3. Applications to Games

In MAZE (see [2]), the board is a rectangular array with walls and a token. See the picture below. Left moves the token down as many squares as she likes without passing over a wall; Right moves to the right with the same constraint. The first player who cannot move is the loser. Very little is known about MAZE. All the other games considered in this section have the property that the opponent's 'before and after' options are related. Left's options after a move by Right in MAZE need not have any relation to those before the move. In the following example, Right's moves are to $\{3 \mid -1\}$, -1, $\{1 \mid 0\}$ and 0 whereas, Left moving first, moves to 2, 1 or 0.



We can say much more about ROLL THE LAWN and CRICKET PITCH and this we do in the next subsections.

3.1. ROLL THE LAWN. The main observation is that if the roller starts the game to the right, say, of a particular bump then at the end of the game, regardless of the actual sequence of moves, it will finish on the right side of the bump if the bump is even and on the left side if the bump is odd. The winner is the player that has more odd bumps in their direction. To see this more mathematically.

Lemma 3.1. *Let* $G = [a_1, a_2, \ldots, a_j, \ominus, a_{j+1}, \ldots, a_k]$ *be a* ROLL THE LAWN *position and let* $b_i = a_i \pmod 2$ *then*

$$[a_1, a_2, \ldots, a_j, \ominus, a_{j+1}, \ldots, a_k] = [b_1, b_2, \ldots, b_j, \ominus, b_{j+1}, \ldots, b_k].$$

Proof. We leave it to the reader to show that

$$[a_1, a_2, \ldots, a_j, \ominus, a_{j+1}, \ldots, a_k] - [b_1, b_2, \ldots, b_j, \ominus, b_{j+1}, \ldots, b_k]$$

is a second player win. □

It is an easy to proof that the actual value is a number and is given by the next result.

Corollary 3.2. *Let* $G = [a_1, a_2, \ldots, a_j, \ominus, a_{j+1}, \ldots, a_k]$ *be a* ROLL THE LAWN *position and let* $b_i = a_i \pmod 2$ *then*

$$G = \sum_{i \le j} b_i - \sum_{i > j} b_i.$$

Consequently, our original example of $[5, \ominus, 2, 4]$ has value $1 - 0 = 1$.

3.2. CRICKET PITCH. For brevity, if $X = (x_i)_{i=1}^n$ then we put $X - 1 = (x_i - 1)_{i=1}^n$. In the analysis of the CRICKET PITCH position $[X, \ominus, Y]$ where X and Y are strings of non-negative integers, a typical left move will be to $[X_1, \ominus, X_2 - 1, Y]$. Our original example $[5, \ominus, 1, 2, 4]$ has value $-1/2$, other values that occur are

$$[1, \ominus, 1] \quad = \quad *,$$

$$\begin{aligned}
[1,3,\ominus,3,1] &= *2,\\
[1,3,5,\ominus,5,3,1] &= *3,\\
[1,3,5,7,\ominus,7,5,3,1] &= *4,\\
[1,1,\ominus,3,3,2] &= 5/4,\\
[1,2,\ominus,3,3,1] &= \downarrow+\downarrow^2*,\\
[1,1,\ominus,3,3,1] &= \{0,\{0,\{0|-1\}|-1\}|-1\},\\
[1,3,\ominus,3,1,2] &= *2,\\
[1,2,\ominus,3,3,3] &= 1/8.
\end{aligned}$$

Since CRICKET PITCH is an option-closed game, then the reduced canonical form for $[X,\ominus,Y]$ is either a number or a switch $\{a|b\}$.

Observation 3.3. *Note that if the position has a 0-bump then anything on the other side of the bump from the roller is irrelevant since the roller can never reach it, and so can be dropped from the positon without changing the game. That is*

$$[2,3,0,3,4,\ominus,1,2,0,56] = [3,4,\ominus,1,2]$$

Lemma 3.4.

$$[a_1,a_2,\dots,a_j,\ominus,a_{j+1},\dots,a_k] = [a_1+2,a_2+2,\dots,a_j+2,\ominus,a_{j+1}+2,\dots,a_k+2]$$

Proof. Let A, B be sequences of non-negative numbers. We let $(A+2)$ denote the sequence where every number in A has been increased by 2. Consider the position $[A,\ominus,B]$ and consider the game $[(A+2),\ominus,(B+2)]-[A,\ominus,B]$ with Left moving first. We claim that Left cannot win.

If Left moves to

$$[(A+2),\ominus,(B+2)]-[A,B',\ominus,B'']$$

then Right responds with the 'mirror' move in the other component to

$$[(A+2),(B'+1),\ominus,(B''+2)]-[A,B',\ominus,B'']$$

which is 0 by induction.

If Left moves to

$$[(A'+2),\ominus,(A''+1),(B+2)]-[A,\ominus,B]$$

then if possible, Right mirrors by playing to

$$[(A'+2),\ominus,(A''+1),(B+2)]-[A',\ominus,(A''-1),B]$$

which is zero by induction. If this move is not possible then we know that Left moved over a 2 in $(A+2)$. That is, $A = C,0,D$ where D does not contain any 0s. Then Left initially moved

$$[(C+2),2,(D+2),\ominus,(B+2)]-[C,0,D,\ominus,B]$$
$$\rightarrow [(C'+2),\ominus,(C''+1),1,(D+1),(B+2)]-[C,0,D,\ominus,B].$$

We claim that Right wins by moving to

$$[(C'+2)\ominus,(C''+1),1,(D+1),(B+2)]-[C,0,\ominus,(D-1),B].$$

From here, if Left moves in the second component of

$$[(C'+2)\ominus,(C''+1),1,(D+1),(B+2)]-[C,0,\ominus,(D-1),B]$$

Right mirrors Left's move in the other component and the resulting position is 0 by induction. Otherwise, Left moves in $[(C'+2)\ominus,(C''+1),1,(D+1),(B+2)]$ then Right responds in the same component moving over but not further than the bump of size 1, i.e. ignoring the bumps that are no longer reachable. He moves to

$$[0,\ominus,(D+1),(B+2)]-[0,\ominus,(D-1),B]$$

and this is 0 by induction. □

Definition 3.5. *Let G be the Cricket Pitch position $A\ominus B$. The Left odd, low point, ldip(G), is*

$$ldip(G)=\min\{a_i|a_i \text{ is odd and } a_i<a_j, i<j\},$$

If there is no such heap then ldip$(G)=\infty$. Similarly,

$$rdip(G)=\min\{b_i|b_i \text{ is odd and } b_j>b_i, i>j\},$$

For example, in $[1,2,3,4,\ominus,1,2,2,3]$, the heaps in position 1 and 3 are both smaller than the heaps between themselves and the roller but the heap of size 1 is the least so that ldip$(G)=1$. Note that rdip$(G)=1$ since the heap immediately to the right of the roller is odd and there is no smaller heap between it and the roller. In $[3,2,3,4,\ominus,2,2,2,3]$ ldip$(G)=3$ and rdip$(G)=\infty$.

The previous Lemma and Observation 3.3 give an algorithm to simplify CRICKET PITCH positions. If we apply Lemma 3.4 repeatedly, we eventually reach an equivalent position with a heap of size either 0 or 1, we say that this position has been *reduced*. From Observation 3.3, if a position has a 0 heap, it can be replaced by a simpler, we say *pruned*, position. A position has been *reduced and pruned* if these two operations are repeated until either there are no heaps left, or there is a heap of size 1 but none of size 0. For example:

$$\begin{aligned}[3,2,5,8,\ominus,6,2,5,7] &\rightarrow [1,0,3,6,\ominus,4,0,3,5] \text{ reduce}\\ &\rightarrow [3,6,\ominus,4] \text{ prune}\\ &\rightarrow [1,4,\ominus,2] \text{ reduce}\end{aligned}$$

Theorem 3.6. *Let $G=A,\ominus,B$ be Cricket Pitch position. The outcome classes are determined by the odd low points:*

1. *if ldip$(G)<$ rdip(G) then $G\in\mathcal{L}$;*
2. *if ldip$(G)>$ rdip(G) then $G\in\mathcal{R}$;*
3. *if ldip$(G)=$ rdip$(G)<\infty$ then $G\in\mathcal{N}$;*
4. *if ldip$(G)=$ rdip$(G)=\infty$ then $G\in\mathcal{P}$.*

Proof. For part 1, since $\mathrm{ldip}(G)$ exists it is an odd number and is also smaller than all the heaps between it and the roller, therefore we can reduce and prune to get an equivalent position G' with $\mathrm{ldip}(G') = 1$, $\mathrm{rdip}(G') > 1$ and no heap of size 1 to the right of the roller. Now Left can win G', and so G, playing first or second since she can always move past the 1 leaving Right with no move.

A similar argument holds for part 2.

For part 3, the reduced-and-pruned G' will have $\mathrm{ldip}(G) = \mathrm{rdip}(G) = 1$ and now both players have a winning first move.

Suppose $\mathrm{ldip}(G) = \mathrm{rdip}(G) = \infty$. Now any odd-sized heap has at least one smaller even-sized heap between it and the roller then all odd heaps will get pruned. Therefore the reduced-and-pruned position is $\ominus$ with no heaps. □

Intuitively, low odd numbers and high even numbers are preferred. We can make this statement more exact.

For playing CRICKET PITCH in isolation, Theorem 3.6 is good enough. But how does one play a combination of several games of CRICKET PITCH, or CRICKET PITCH with other games? For example, in the following disjunctive sum where every position is in $\mathcal{N}$

$$[1,2,3,1,\ominus,4,1,3] + [1,1,3,2,\ominus,1,3,1] \\ +[3,3,2,1,2,\ominus,2,2,1] + [1,2,\ominus,3,1,2]$$

but who wins? The next result is a step in the right direction.

Theorem 3.7. *Let G be a* CRICKET PITCH *position.*

- *If $\infty = ldip(G) = rdip(G)$ then $G = 0$;*
- *If $ldip(G) = rdip(G)$ then $\overline{G} = \{a \mid b\}$ for integers a and b, $a \geq 0 \geq b$ or $\overline{G} = 0$ if $a = b = 0$.*

Proof. Case 1 follows since $G \in \mathcal{P}$.

If $\mathrm{ldip}(G) = \mathrm{rdip}(G)$ then the reduced-pruned game has a heaps of size 1 to the left and to the right of the roller. Let a be the non-negative integer obtained by Left moving just past the closest heap of size 1 and b the non-positive obtained by Right moving just past the closest heap of size 1.

On Left's turn, she must move at least far enough to go past the closest heap of size 1, otherwise Right will win on her next turn. Also, we know that $\mathrm{LS}(G) \geq a$ since Left has an option to a. If $\mathrm{LS}(G) = x > a$ then it must correspond to a move beyond his move to a. This is a contradiction since it would imply that there is a left option of a that is to a number greater than a. □

4. QUESTIONS AND FURTHER WORK.

From the work in the previous section, we have the following two questions

Question. MAZE *is inherently 2-dimensional. Does this restrict which dyadic rationals that can appear in the reduced canonical form? Are the bounds on* $G - \overline{G}$ *tighter than* $\Uparrow *$*?*

Question. *In* CRICKET PITCH*, if* $ldip(G) < rdip(G)$*, when is* $\overline{G}$ *a number and when is it a switch?*

There are other games are close variants of option-closed games. For example, END NIM [1]; PARTIAL NIM, [8], (played on one heap) is option-closed for one-player but not the other; also in both KONANE, [6] and TOPPLING DOMINOES, [2], the moves have a linear directional aspect. *Does the reduced canonical form help simplify these games?*

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