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A NEW EXAMPLE OF NON-AMORPHOUS ASSOCIATION SCHEMES

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Dedicated to Professor Tatsuro Ito on the occasion of his sixtieth birthday

ABSTRACT. E. R. van Dam gave an example of primitive non-amorphous association schemes in which every nontrivial relation is a strongly regular graph, as a fusion scheme of the cyclotomic scheme of class 45 on $\mathrm{GF}(2^{12})$. The aim of this paper is to present a new example of primitive non-amorphous association schemes in which every nontrivial relation is a strongly regular graph, as a fusion scheme of the cyclotomic scheme of class 75 on $\mathrm{GF}(2^{20})$. We also propose an infinite family of parameters of association schemes containing both of these two examples.

1. INTRODUCTION

Let X be a finite set with cardinality n. Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme of class d on X. Let $P = (p_{i,j})$ and $Q = (q_{i,j})$ (where $0 \le i, j \le d$) be the first and the second eigenmatrices of $(X, \{R_i\}_{i=0}^d)$ respectively. We refer [2] for notation and general theory of association schemes.

Let $\{\Lambda_j\}_{j=0}^{d'}$ be a partition of $\{0, 1, \ldots, d\}$ with $\Lambda_0 = \{0\}$. We define $R_{\Lambda_j} = \bigcup_{\ell \in \Lambda_j} R_\ell$. If $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ forms an association scheme, then we call $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ a *fusion* scheme of $(X, \{R_i\}_{i=0}^d)$. If $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ is an association scheme for any partition $\{\Lambda_j\}_{j=0}^{d'}$ of $\{0, 1, \ldots, d\}$ with $\Lambda_0 = \{0\}$, then $(X, \{R_i\}_{i=0}^d)$ is called *amorphous*.

There is a simple criterion in terms of P for a given partition $\{\Lambda_j\}_{j=0}^{d'}$ to give rise to a fusion scheme (due to Bannai [1], Muzychuk [9]): There exists a partition $\{\Delta_i\}_{i=0}^{d'}$ of $\{0, 1, \ldots, d\}$ with $\Delta_0 = \{0\}$ such that each (Δ_i, Λ_j)-block of the first eigenmatrix P has a constant row sum. The constant row sum turns out to be the (i, j) entry of the first eigenmatrix of the fusion scheme.

Let q be a prime power, and e be a divisor of $q - 1$. Fix a primitive element α of the multiplicative group of the finite field $\mathrm{GF}(q)$. Then $\langle \alpha^e \rangle$ is

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a subgroup of index e and its cosets are $\alpha^i\langle\alpha^e\rangle$ $(0 \le i \le e-1)$. We define $R_0 = \{(x,x)|x \in \mathrm{GF}(q)\}$ and $R_i = \{(x,y)|x-y \in \alpha^i\langle\alpha^e\rangle, x,y \in \mathrm{GF}(q)\}$ $(1 \le i \le e)$. Then $(\mathrm{GF}(q), \{R_i\}_{i=0}^e)$ forms an association scheme and is called the *cyclotomic* scheme of class e on $\mathrm{GF}(q)$.

The cyclotomic scheme of class e on $\mathrm{GF}(q)$ is symmetric if and only if q or $(q-1)/e$ is even. Moreover, in this case, it is amorphous if and only if -1 is a power of p modulo e, where p is the characteristic of $\mathrm{GF}(q)$. Although formulated differently, this result is due to Baumert, Mills and Ward [3] (see also [5]). Ito, Munemasa and Yamada constructed amorphous association schemes over Galois rings. Clearly, in an amorphous association scheme, every nontrivial relation is a strongly regular graph. A. V. Ivanov [8] conjectured the converse also holds, but later it was disproved by van Dam [10]. Since the counterexample given in [10] was an imprimitive association scheme, it remained as an unsolved problem to find a primitive non-amorphous association scheme in which every nontrivial relation is a strongly regular graph. In [11], van Dam constructed a non-amorphous 4-class fusion scheme of the cyclotomic scheme of class 45 on $\mathrm{GF}(2^{12})$ with the following first eigenmatrix:

$$\begin{pmatrix} 1 & 3276 & 273 & 273 & 273 \\ 1 & -52 & 17 & 17 & 17 \\ 1 & 12 & -15 & -15 & 17 \\ 1 & 12 & -15 & 17 & -15 \\ 1 & 12 & 17 & -15 & -15 \end{pmatrix}. \tag{1.1}$$

This was the first and the only known primitive non-amorphous association scheme in which every nontrivial relation is a strongly regular graph.

In this paper, we present another such example.

Theorem 1.1. *The cyclotomic scheme of class* 75 *on* $\mathrm{GF}(2^{20})$ *has a non-amorphous fusion scheme of class* 4 *with the following first eigenmatrix :*

$$\begin{pmatrix} 1 & 838860 & 69905 & 69905 & 69905 \\ 1 & -820 & 273 & 273 & 273 \\ 1 & 204 & -239 & -239 & 273 \\ 1 & 204 & -239 & 273 & -239 \\ 1 & 204 & 273 & -239 & -239 \end{pmatrix}. \tag{1.2}$$

2. Restrictions on the first eigenmatrix

In general, if an association scheme $(X, \{R_i\}_{i=0}^d)$ has the following first eigenmatrix (2.1), then for each relation R_i $(i = 1,2,3,4)$, (X, R_i) is a strongly regular graph, and $(X, \{R_i\}_{i=0}^d)$ is not amorphous.

$$P = \left(\begin{array}{cc|ccc} 1 & k_1 & k_2 & k_2 & k_2 \\ 1 & s_1 & r_2 & r_2 & r_2 \\ \hline 1 & r_1 & s_2 & s_2 & r_2 \\ 1 & r_1 & s_2 & r_2 & s_2 \\ 1 & r_1 & r_2 & s_2 & s_2 \end{array}\right). \tag{2.1}$$

Indeed, clearly $r_2 \neq s_2$, so $(X, \{R_0, R_1 \cup R_2, R_3 \cup R_4\})$ is not an association scheme.

Lemma 2.1. *Let $(X, \{R_i\}_{i=0}^4)$ be an association scheme with the first eigenmatrix* (2.1)*. Then r_1, s_1, r_2, s_2 are integers, and all the parameters can be expressed in terms of r_1, s_1 as follows:*

$$|X| = \frac{(r_1 - s_1)^2(s_1 + 4)}{4(s_1 + 3r_1 + 4)}, \tag{2.2}$$

$$k_1 = \frac{r_1(r_1 s_1 + 4r_1 - s_1^2 + 4)}{s_1 + 3r_1 + 4}, \tag{2.3}$$

$$k_2 = -\frac{r_1 s_1 + 4r_1 - s_1^2 + 4}{12} \tag{2.4}$$

$$r_2 = -\frac{1}{3}(s_1 + 1), \tag{2.5}$$

$$s_2 = \frac{-3r_1 + s_1 - 2}{6}, \tag{2.6}$$

$$m_1 = \frac{1}{4}k_1, \tag{2.7}$$

$$m_2 = m_3 = m_4 = -\frac{s_1 + 4}{12r_1}k_1. \tag{2.8}$$

Proof. By [2, Chap.2, Theorem 4.1], we have $m_2 = m_3 = m_4$. By [2, Chap.2, Theorem 3.5], the second eigenmatrix Q is given by

$$Q = \begin{pmatrix} 1 & m_1 & m_2 & m_2 & m_2 \\ 1 & \frac{s_1 m_1}{k_1} & \frac{r_1 m_2}{k_1} & \frac{r_1 m_2}{k_1} & \frac{r_1 m_2}{k_1} \\ 1 & \frac{r_2 m_1}{k_2} & \frac{s_2 m_2}{k_2} & \frac{s_2 m_2}{k_2} & \frac{r_2 m_2}{k_2} \\ 1 & \frac{r_2 m_1}{k_2} & \frac{s_2 m_2}{k_2} & \frac{r_2 m_2}{k_2} & \frac{s_2 m_2}{k_2} \\ 1 & \frac{r_2 m_1}{k_2} & \frac{r_2 m_2}{k_2} & \frac{s_2 m_2}{k_2} & \frac{s_2 m_2}{k_2} \end{pmatrix}. \tag{2.9}$$

Since $PQ = |X|I$, we have

$$0 = (PQ)_{1,0} = 1 + s_1 + 3r_2,$$
$$0 = (PQ)_{2,0} = 1 + r_1 + 2s_2 + r_2.$$

These give (2.5) and (2.6). Also, we have

$$0 = (PQ)_{1,2} = m_1 \left(1 + \frac{r_1 s_1}{k_1} + \frac{2r_2 s_2 + r_2^2}{k_2}\right),$$
$$0 = (PQ)_{2,3} = m_2 \left(1 + \frac{r_1^2}{k_1} + \frac{s_2^2 + 2r_2 s_2}{k_2}\right).$$

Thus

$$\begin{pmatrix}\frac{1}{k_1} & \frac{1}{k_2}\end{pmatrix}\begin{pmatrix} r_1 s_1 & r_1^2 \\ 2r_2s_2 + r_2^2 & s_2^2 + 2r_2s_2\end{pmatrix} = \begin{pmatrix}-1 & -1\end{pmatrix}.$$

Solving this and substituting (2.5) and (2.6), we obtain (2.3) and (2.4). Now, (2.2), (2.7) and (2.8) follow from

$$|X| = (PQ)_{0,0} = 1 + k_1 + 3k_2,$$
$$|X| = (PQ)_{1,1} = m_1\left(1 + \frac{s_1^2}{k_1} + 3\frac{r_2^2}{k_2}\right),$$
$$|X| = (PQ)_{2,2} = m_2\left(1 + \frac{r_1^2}{k_1} + 2\frac{s_2^2}{k_2} + \frac{r_2^2}{k_2}\right).$$

Finally, we show that r_1 and s_1 are integers. Since they are algebraic integers, it suffices to show that they are rational. Suppose to the contrary. Then by (2.5), r_2 is irrational, so r_2 and s_2 are algebraic conjugate. Thus $m_1 + m_4 = m_2 + m_3$, as the multiplicities of r_2 and s_2 in (X, R_2) are equal. By (2.8), this implies $m_1 = m_2$. On the other hand, the same argument applied to (X, R_1) implies $m_1 = 3m_2$, which is a contradiction. □

Recall that a symmetric association scheme $(X, \{R_i\}_{i=0}^d)$ is formally self-dual if its first eigenmatrix P coincides with its second eigenmatrix, after permuting the rows and the columns of P. We say that a strongly regular graph (X, R) is formally self-dual if the associated association scheme of class 2 is formally self-dual. Note that, we can see easily from (2.7) and (2.9) that any association scheme with the first eigenmatrix (2.1) is not formally self-dual. In van Dam's example with the first eigenmatrix (1.1), however, the strongly regular graph (X, R_1) is formally self-dual. If we adopt this as an assumption, then we have a one-parameter family of possible first eigenmatrices:

Lemma 2.2. *Let $(X, \{R_i\}_{i=0}^4)$ be an association scheme with the first eigenmatrix (2.1), and assume that the strongly regular graph (X, R_1) is formally self-dual. Then $s_1 = -4r_1 - 4$, $r_1 \equiv 0 \pmod 6$ and $|X| = (5r_1 + 4)^2$.*

Proof. By the assumption, $k_1 \in \{m_1, |X| - m_1 - 1\}$. By (2.7) and (2.8), we obtain $s_1 = -4r_1 - 4$. Then by (2.2), we obtain $|X| = (5r_1 + 4)^2$. Also by (2.6), $s_2 = (-7r_1 - 6)/6$, and hence $r_1 \equiv 0 \pmod 6$. □

Setting $r = \frac{r_1}{6}$, the first eigenmatrix of an association scheme satisfying the hypotheses of Lemma 2.2 has the following form:

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_2 & k_2 \\ 1 & -4(6r+1) & 8r+1 & 8r+1 & 8r+1 \\ 1 & 6r & -7r-1 & -7r-1 & 8r+1 \\ 1 & 6r & -7r-1 & 8r+1 & -7r-1 \\ 1 & 6r & 8r+1 & -7r-1 & -7r-1 \end{pmatrix}, \tag{2.10}$$

where $k_1 = 12(6r + 1)(10r + 1)$ and $k_2 = (6r + 1)(10r + 1)$.

3. Construction of a new example

We consider the problem of realizing (2.10) as the first eigenmatrix of a cyclotomic association scheme. By Lemma 2.2, $|X|$ is even, so we assume $|X|$ is a power of 2. Put $30r+4=\sqrt{|X|}=2^g$. Then $2^g \equiv 4 \pmod 5$, and hence $g=4h+2$ for some nonnegative integer h. In this case, $|X|=2^{8h+4}$ and $r=\frac{2}{15}(16^h-1)$.

When $h=0$, we have

$$P=\begin{pmatrix} 1 & 12 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

This is realized as the first eigenmatrix of an association scheme belonging to an infinite family of imprimitive non-amorphous association schemes appeared in [10], and it is also mentioned in [11].

The case $h=1$ gives the matrix (1.1) which was realized in [11].

When $h=2$, we obtain the matrix (1.2). This is realized as a fusion scheme of the cyclotomic scheme of class 75 on $\mathrm{GF}(2^{20})$. Let α be a primitive element satisfying

$$\alpha^{20}+\alpha^{10}+\alpha^9+\alpha^7+\alpha^6+\alpha^5+\alpha^4+\alpha+1=0.$$

Let

$$H_j=\{(x,y)\mid x-y\in\alpha^j\langle\alpha^{75}\rangle\}\quad (j=0,1,\dots,74)$$

By computer, we have verified that the graph Γ on $\mathrm{GF}(2^{20})$ with edge set

$$R_2=H_0\cup H_3\cup H_6\cup H_9\cup H_{12}$$

is a strongly regular graph with eigenvalues $69905, 273, -239$. Clearly, each of the graphs with edge sets

$$R_3=H_{25}\cup H_{28}\cup H_{31}\cup H_{34}\cup H_{37},$$
$$R_4=H_{50}\cup H_{53}\cup H_{56}\cup H_{59}\cup H_{62}$$

are isomorphic to Γ. Moreover, since $H_0\cup H_{25}\cup H_{50}$ is one of the relations in the 25-class cyclotomic amorphous association scheme on $\mathrm{GF}(2^{20})$, the union $R_2\cup R_3\cup R_4$ is a strongly regular graph with eigenvalues $209715, 819, -205$, by [5, Theorem 2]. Hence the complement Γ_1 of this union is strongly regular with eigenvalues $838860, 204, -820$. Let R_0 denote the diagonal relation on $\mathrm{GF}(2^{20})$, and let R_1 denote the edge set of Γ_1. Then the association scheme $(\mathrm{GF}(2^{20}),\{R_i\}_{i=0}^4)$ has the character table as described in (1.2). This completes the proof of Theorem 1.1.

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