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REPRESENTATION FOR SOME ALGEBRAS WITH A NEGATION OPERATOR

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Abstract. In [1], the variety of $\neg$-lattices was introduced as a bounded distributive lattices, $\boldsymbol{A}$, endowed with a unary operation $\neg$, satisfying the axioms $\neg 0 \approx 1$ and $\neg(a \vee b) \approx \neg a \wedge \neg b$. In this paper, we apply the representation and the duality developed in [1] to give short representations for semi-De Morgan algebras, demi-p-lattices, almost p-lattices, and weak Stone algebras.

1. Introduction and preliminaries

In [1], the variety $\mathcal{N}$ of $\neg$-lattices was introduced as a generalization of some known algebraic structures that have, as a reduct, a bounded distributive lattice, and are endowed with a unary operation $\neg$. Examples include p-algebras [2], semi-De Morgan algebras, demi-p-lattices and almost p-lattices [9], and quasi-Stone algebras [1].

For all these varieties there are representation theorems and Priestley dualities. In the case of the variety of semi-De Morgan algebras, a duality is given in [5]; for the variety of demi-p-lattices and the variety of almost p-lattices a duality is presented in [3] (see also [7]); for the variety of quasi-Stone algebras, a Priestley duality is developed in [1]. In this note, we use the results rendered in [1] to give alternative and short proofs of the main results on the representation for semi-De Morgan algebras, demi-p-lattices and almost p-lattices. We also give a representation for weak-Stone algebras [9].

An algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a distributive lattice with a *negation operator* $\neg$ (or $\neg$-lattice), if $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and the unary operation $\neg$ satisfies the identities:

(1) $\neg 0 \approx 1$;
(2) $\neg(x \vee y) \approx \neg x \wedge \neg y$.

The varieties of semi-De Morgan algebras $\mathcal{SDMA}$, the variety of demi-p-lattices $\mathcal{DMPL}$, the variety of almost p-lattices $\mathcal{ADPL}$, and the variety

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$\mathcal{WS}$ of weak-Stone algebras are important examples of $\neg$-lattices, and can be axiomatized as follows.

$$\begin{aligned}
\mathcal{SDMA} &= \mathcal{N}+\{\neg 1\approx 0, \neg\neg(x\wedge y)\approx\neg\neg x\wedge\neg\neg y,\ \neg\neg\neg x\approx\neg x\}.\\
\mathcal{DMPL} &= \mathcal{SDMA}+\{\neg x\wedge\neg\neg x\approx 0\}.\\
\mathcal{ADPL} &= \mathcal{SDMA}+\{\neg x\wedge x\approx 0\}.\\
\mathcal{WS} &= \mathcal{SDMA}+\{\neg x\vee\neg\neg x\approx 1\}.
\end{aligned}$$

A characterization of the variety of semi-De Morgan algebras is given in the following result.

Lemma 1.1. *Let $\boldsymbol{A}\in\mathcal{N}+\{\neg 1\approx 0\}$. Then the following conditions are equivalent.*

(1) $\boldsymbol{A}\in\mathcal{SDMA}$.
(2) $\boldsymbol{A}\vDash\neg(a\wedge b)\approx\neg(a\wedge\neg\neg b)$.

Proof. (1 $\Rightarrow$ 2) Let $a,b\in A$. Then

$$\begin{aligned}
\neg(a\wedge b) &= \neg\neg\neg(a\wedge b) &&= \neg(\neg\neg a\wedge\neg\neg b)\\
&= \neg(\neg\neg a\wedge\neg\neg\neg\neg b) &&= \neg\neg\neg(a\wedge\neg\neg b)\\
&= \neg(a\wedge\neg\neg b).
\end{aligned}$$

(2 $\Rightarrow$ 1) Let $a,b\in A$. Then

$$\neg a = \neg(1\wedge a) = \neg(1\wedge\neg\neg a) = \neg\neg\neg a$$

On the other hand,

$$\begin{aligned}
\neg\neg(a\wedge b) &= \neg(\neg(a\wedge b)) &&= \neg(\neg(a\wedge\neg\neg b))\\
&= \neg(\neg(\neg\neg a\wedge\neg\neg b)) &&= \neg\neg(\neg\neg a\wedge\neg\neg b)\\
&= \neg\neg\neg(\neg a\vee\neg b) &&= \neg(1\wedge\neg\neg(\neg a\vee\neg b))\\
&= \neg(1\wedge(\neg a\vee\neg b)) &&= \neg(\neg a\vee\neg b)\\
&= \neg\neg a\wedge\neg\neg b.
\end{aligned}$$

Therefore, $\boldsymbol{A}\in\mathcal{SDMA}$. □

2. Representation

First we describe the representation of $\neg$-lattices by means of suitable ordered sets equipped with binary relations. Given a poset $\langle X,\leq\rangle$, a set $Y\subseteq X$ is increasing if it closed under $\leq$, that is, if for every $x\in Y$ and every $y\in X$, if $x\leq y$ then $y\in Y$. The set of all increasing subsets of X will be denoted by $\mathcal{P}_u(X)$, and the power set of X by $\mathcal{P}(X)$. If R is a binary relation on a set X and $x\in X$, we define $R(x)=\{y\in X: xRy\}$. An ordered $\neg$-*frame,* or frame for short, is a relational structure $\langle X,\leq,R\rangle$ where $\langle X,\leq\rangle$ is a poset and R is a binary relation on X such that $(\leq\circ R\circ\leq^{-1})\subseteq R$. In any frame $\langle X,\leq,R\rangle$ the set $\mathcal{P}_u(X)$ is closed under the operation $\neg_R$ defined by:

$$\neg_R(U)=\{x\in X: R(x)\cap U=\varnothing\},$$

for each $U \in \mathcal{P}_u(X)$. It is easy to see that the structure

$$A(\mathcal{F}) = \langle \mathcal{P}_u(X), \cup, \cap, \neg_R, \varnothing, X \rangle$$

is a $\neg$-lattice.

Let $\boldsymbol{A}$ be a bounded distributive lattice. The filter (ideal) generated by a set $H \subseteq A$ is denoted by $[H)$ ($(H]$). The set of all prime filters of a $\neg$-lattice is denoted by $X(\boldsymbol{A})$. The set of all maximal elements (with respect to $\subseteq$) of $X(\boldsymbol{A})$ is denoted by $\max X(\boldsymbol{A})$. For a subset $Y \neq \varnothing$ of $X(\boldsymbol{A})$, let $\max Y = \{x \in Y : x \leq y \text{ and } y \in Y, \text{implies } x = y\}$.

We define a binary relation $R_\neg$ on the set $X(\boldsymbol{A})$. Let $R_\neg \subseteq X(\boldsymbol{A}) \times X(\boldsymbol{A})$ given by

$$(P, Q) \in R_\neg \text{ if and only if } \neg^{-1}(P) \cap Q = \varnothing.$$

We also define a relation $R_m \subseteq X(\boldsymbol{A}) \times X(\boldsymbol{A})$ as

$$(P, Q) \in R_m \text{ if and only if } Q \in \max R_\neg(P). \tag{2.1}$$

Lemma 2.1. [1] *Let $\boldsymbol{A} \in \mathcal{N}$, and $P \in X(\boldsymbol{A})$. Then*

1. *$\neg^{-1}(P) = \{a \in A \mid \neg a \in P\}$ is an ideal of $\boldsymbol{A}$;*
2. *for each $a \in A$, $\neg a \notin P$ if and only if there is $Q \in X(\boldsymbol{A})$ such that $(P, Q) \in R_m$ and $a \in Q$.*

Let us consider the structure $\mathcal{F}(\boldsymbol{A}) = \langle X(\boldsymbol{A}), \subseteq, R_\neg \rangle$. It is easy to see that $\mathcal{F}(\boldsymbol{A})$ is a $\neg$-frame because $\left(\subseteq \circ R_\neg \circ \subseteq^{-1}\right) \subseteq R_\neg$. Then

$$E_c(\boldsymbol{A}) = \left\langle \mathcal{P}_u(X(\boldsymbol{A})), \cup, \cap, \neg_{R_\neg}, \varnothing, X(\boldsymbol{A}) \right\rangle$$

is a $\neg$-lattice called the *canonical extension* of $\boldsymbol{A}$. As in the case of the representation for bounded distributive lattices, to obtain a representation theorem for $\neg$-lattices, we consider the family of sets $\sigma(\boldsymbol{A}) = \{\sigma(a) : a \in A\}$, where for each $a \in A$, $\sigma(a) = \{P \in X(\boldsymbol{A}) : a \in P\}$. Then it is easy to see that the set $\sigma(\boldsymbol{A})$ is closed under the operation $\neg_{R_\neg}$ defined on $\mathcal{P}_u(X(\boldsymbol{A}))$. Thus, the algebra

$$\langle \sigma(\boldsymbol{A}), \cup, \cap, \neg_{R_\neg}, \varnothing, X(\boldsymbol{A}) \rangle$$

is a subalgebra of the canonical extension $E_c(\boldsymbol{A})$ of $\boldsymbol{A}$. Then we have that every $\neg$-lattice $\boldsymbol{A}$ is isomorphic to the $\neg$-lattice of sets $\sigma(\boldsymbol{A})$, that is, σ is an embedding of $\boldsymbol{A}$ into $E_c(\boldsymbol{A})$.

Theorem 2.2. *Let $\boldsymbol{A} \in \mathcal{N}$. Then*

1. *$\boldsymbol{A} \vDash \neg a \wedge a \approx 0 \Leftrightarrow R_\neg$ is reflexive.*
2. *$\boldsymbol{A} \vDash \neg 1 \approx 0 \Leftrightarrow R_\neg$ is serial i.e., $R_\neg(P) \neq \varnothing$ for any $P \in X(\boldsymbol{A})$.*

Proof. We prove only (1). Suppose that $\neg a \wedge a \approx 0$ is valid in $\boldsymbol{A}$. Let $P \in X(\boldsymbol{A})$. Then for all $a \in P$, $\neg a \notin P$. Thus, $(P, P) \in R_\neg$. Conversely, if there exists $a \in A$ such that $\neg a \wedge a \neq 0$, then there exists $P \in X(\boldsymbol{A})$ such that $\neg a \in P$ and $a \in P$, which is a contradiction. Thus, $\neg a \wedge a = 0$ for every $a \in A$. ∎

Consider the set

$$X_m = \bigcup \{\max R_\neg(P) : P \in X(\boldsymbol{A})\}. \tag{2.2}$$

For each $P \in X(\boldsymbol{A})$, define the set

$$g(P) = \{a \in A : \neg a \notin P\}.$$

Now we give the characterization of semi-De Morgan algebras in terms of the relation R_m.

Theorem 2.3. *Let $\boldsymbol{A} \in \mathcal{N}$. Then the following conditions are equivalent.*

1. *$\boldsymbol{A} \in \mathcal{SDMA}$.*
2. *For any $Q \in X_m$, $g^2(Q) = Q$, $g(Q) \in X_m$, $R_m(Q) = \{g(Q)\}$, and $R_m(g(Q)) = \{Q\}$. Thus, g is an involution on X_m.*
3. *The relation R_m is serial, and for any $Q, D, Z \in X_m$, if $(Q, D) \in R_m$ and $(D, Z) \in R_m$, then $Q = Z$.*

Proof. ($1 \Rightarrow 2$) Let $Q \in X_m$. We prove that $a \in Q$ if and only if $\neg\neg a \in Q$. Suppose that $a \in Q$, and $\neg\neg a \notin Q$. Then there exists $P \in X(\boldsymbol{A})$ such that $(P, Q) \in R_m$, and

$$\neg^{-1}(P) \cap [Q \cup \{\neg\neg a\}) \neq \varnothing,$$

i.e., there exists $q \in Q$ such that $\neg(q \wedge \neg\neg a) \in P$. From Lemma 1.1, $\neg(q \wedge \neg\neg a) = \neg(q \wedge a)$, so

$$q \wedge a \in \neg^{-1}(P) \cap Q,$$

a contradiction, because $(P, Q) \in R_\neg$. Thus, $\neg\neg a \in Q$. Similarly we can prove that if $\neg\neg a \in Q$, then $a \in Q$. Thus, $g^2(Q) = Q$.

We prove that $g(Q)$ is a prime filter. It is clear that $g(Q)$ is increasing, $1 \in g(Q)$, and that if $a \vee b \in g(Q)$, then $a \in g(Q)$ or $b \in g(Q)$.

We prove that $g(Q)$ is closed under $\wedge$. Let $a, b \in g(Q)$, i.e., $\neg a$, $\neg b \notin Q$. Let $P \in X(\boldsymbol{A})$ such that $(P, Q) \in R_m$. Then,

$$\neg^{-1}(P) \cap [Q \cup \{\neg a\}) \neq \varnothing \text{ and } \neg^{-1}(P) \cap [Q \cup \{\neg b\}) \neq \varnothing,$$

and there exist $q_1, q_2 \in Q$ such that

$$\neg(q_1 \wedge \neg a) \in P \text{ and } \neg(q_2 \wedge \neg b) \in P.$$

Let $q = q_1 \wedge q_2$. Then

$$\begin{aligned}
\neg(q \wedge \neg a) \wedge \neg(q \wedge \neg b) &= \neg((q \wedge \neg a) \vee (q \wedge \neg b)) \\
&= \neg(q \wedge (\neg a \vee \neg b)) \\
&= \neg(q \wedge \neg\neg(\neg a \vee \neg b)) \text{ (by Lemma 1.1)} \\
&= \neg(q \wedge \neg(\neg\neg a \wedge \neg\neg b)) \\
&= \neg(q \wedge \neg\neg\neg(a \wedge b)) \\
&= \neg(q \wedge \neg(a \wedge b)) \in P.
\end{aligned}$$

Since $\neg^{-1}(P) \cap Q = \varnothing$, and $q \in Q$, we get $\neg(a \wedge b) \notin Q$. Thus,

$$a \wedge b \in g(Q),$$

and $g(Q)$ is a prime filter.

It is clear that $(Q, g(Q)) \in R_{\neg}$, and as $g^2(Q) = Q$, we have that $(g(Q), Q) \in R_{\neg}$.We prove that $g(Q) \in R_m(Q)$. Suppose that there exists $D \in R_{\neg}(Q)$ such that $g(Q) \subseteq D$. Then $D \subseteq g(Q)$ and $g(Q) \subseteq D$, i.e., $D = g(Q)$. Thus, $g(Q) \in R_m(Q)$.

Similarly we can prove that $Q \in R_m(g(Q))$.

We prove that $R_m(Q) = \{g(Q)\}$. Suppose that there exists $D \in R_m(Q)$. Then $D \in R_{\neg}(Q)$, i.e., $D \subseteq g(Q)$. As D is maximal in $R_{\neg}(Q)$, and $(Q, g(Q)) \in R_{\neg}$, we have that $D = g(Q)$. Similarly we can prove that $R_m(g(Q)) = \{Q\}$.

$(2 \Rightarrow 3)$ As $g(Q) \in R_m(Q)$ for any $Q \in X_m$, R_m is serial. Let $Q, D, Z \in X_m$ such that $(Q, D) \in R_m$ and $(D, Z) \in R_m$. As $R_m(Q) = \{g(Q)\}$ and $R_m(D) = \{g(D)\}$, $D = g(Q)$ and $g(D) = Z$. Thus $Q = g^2(Q) = g(D) = Z$.

$(3 \Rightarrow 1)$ We prove that for every $Q, D \in X_m$,

$$\text{if } (Q, D) \in R_m, \text{ then } (D, Q) \in R_m. \tag{2.3}$$

Let $Q, D \in X_m$ be such that $(Q, D) \in R_m$. As R_m is serial, there exists $Z \in X_m$ such that $(D, Z) \in R_m$. From hypothesis, we get $Q = Z$, i.e., $(D, Q) \in R_m$. Thus (2.3) is valid.

Since the operation $\neg$ is anti-monotonic, we have that

$$\neg\neg(a \wedge b) \leq \neg\neg a \wedge \neg\neg b,$$

for all $a, b \in A$. Suppose that there exists $a, b \in A$ such that

$$\neg\neg a \wedge \neg\neg b \nleq \neg\neg(a \wedge b).$$

Then there exists $P \in X(\boldsymbol{A})$ such that $\neg\neg a \wedge \neg\neg b \in P$, and $\neg\neg(a \wedge b) \notin P$. From Lemma 2.1, there exists $Q \in X_m$ such that $(P, Q) \in R_m$ and $\neg(a \wedge b) \in Q$. As R_m is serial, there exists $D \in X_m$ such that $(Q, D) \in R_m$, and thus

$$a \wedge b \notin D.$$

By (2.3) we get $(D, Q) \in R_m$. On the other hand, as $\neg\neg a \wedge \neg\neg b \in P$, $\neg a$, $\neg b \notin Q$. So there exist $D_1, D_2 \in X_m$ such that

$$(Q, D_1) \in R_m, (Q, D_2) \in R_m, a \in D_1, \text{ and } b \in D_2.$$

From (2.3) we have $(D_1, Q) \in R_m$, and as $(Q, D) \in R_m$, $D = D_1$. Similarly we can see that $D = D_2$. Thus, $a, b \in D$, which is a contradiction.

Let $a \in A$. We prove that $\neg a \leq \neg\neg\neg a$. If there exists $P \in X(\boldsymbol{A})$ such that $\neg a \in P$ and $\neg\neg\neg a \notin P$, then there exists $Q \in X_m$ such that $(P, Q) \in R_m$, $\neg\neg a \in Q$, and $a \notin Q$. Since R_m is serial, there exists $D \in X_m$ such that $(Q, D) \in R_m$, and by (2.3) $(D, Q) \in R_m$. Since $\neg\neg a \in Q$, $\neg a \notin D$. Then there exists $Z \in X_m$ such that $(D, Z) \in R_m$ and $a \in Z$. By hypothesis we get $Q = Z$, i.e., $a \in Q$, which is a contradiction.

We prove that $\neg\neg\neg a \leq \neg a$. Suppose that $\neg\neg\neg a \nleq \neg a$. Then there exists $P \in X(\boldsymbol{A})$, and there exists $Q \in X_m$ such that

$$\neg\neg\neg a \in P, (P, Q) \in R_m, \text{ and } a \in Q.$$

As $\neg\neg a \notin Q$, there exists $D \in X_m$ such that $(Q, D) \in R_m$ and $\neg a \in D$. From (2.3), $(D, Q) \in R_m$. Thus, $\neg a \notin D$, because $a \in Q$, which is a contradiction. □

Theorem 2.4. *Let $\boldsymbol{A} \in \mathcal{SDMA}$. Then*

1. *$\boldsymbol{A} \in \mathcal{DMPL}$ if and only if $(Q, Q) \in R_m$, for any $Q \in X_m$;*
2. *$\boldsymbol{A} \in \mathcal{ADPL}$ if and only if $X_m = \max X(\boldsymbol{A})$ and $(Q, Q) \in R_m$, for any $Q \in X_m$.*

Proof. (1) We note that for any $Q \in X_m$,

$$(Q, Q) \in R_m \text{ if and only if } g(Q) = Q,$$

so, we will prove that $g(Q) = Q$. Let $a \in Q$. Since $g^2(Q) = Q$, $\neg\neg a \in Q$, and as $\neg a \wedge \neg\neg a = 0$, $\neg a \notin Q$, i.e., $a \in g(Q)$. Thus, $Q \subseteq g(Q)$.

If $a \in g(Q)$, then $\neg a \notin Q$. Suppose that $a \notin Q$. Then, $\neg a \vee a \notin Q$, and since $Q \in R_m(g(Q))$, we get

$$\neg^{-1}(g(Q)) \cap [Q \cup \{\neg a \vee a\}) \neq \varnothing.$$

So there exists $q \in Q$ such that

$$\begin{aligned}
\neg(q \wedge (\neg a \vee a)) &= \neg(q \wedge \neg\neg(\neg a \vee a)) && \text{(by Lemma 1.1)} \\
&= \neg(\neg\neg q \wedge \neg(\neg\neg a \wedge \neg a)) \\
&= \neg(\neg\neg q \wedge \neg 0) \\
&= \neg(\neg\neg q \wedge 1) \\
&= \neg\neg\neg q = \neg q \in g(Q),
\end{aligned}$$

But $\neg q \in g(Q)$ if and only if $\neg\neg q \notin Q$ if and only if $q \notin Q$, a contradiction. Thus $g(Q) \subseteq Q$, and hence $g(Q) = Q$.

Assume that $(Q, Q) \in R_m$, for any $Q \in X_m$. Suppose that there exists $a \in A$ such that

$$\neg a \wedge \neg\neg a \neq 0.$$

Then there exists $P \in X(\boldsymbol{A})$ such that $\neg a \in P$ and $\neg\neg a \in P$. As $R_\neg(P) \neq \varnothing$, because $\neg 1 = 0$, there exists $Q \in X_m$ such that

$$(P, Q) \in R_m,\ a \notin Q \text{ and } \neg a \notin Q.$$

This implies, $a \in g(Q) = Q$, which is absurd. Thus, $\neg a \wedge \neg\neg a = 0$ for any $a \in A$.

(2). If A is an almost p-demi lattice, then $\neg a \wedge \neg\neg a = 0$. Thus, $g(Q) = Q$ for any $Q \in X_m$.

Let $Q \in X_m$. We prove that Q is maximal. Let $a \notin Q$. Then $a \notin g(Q)$, i.e., $\neg a \in Q$. Thu we have to proved that for any $a \notin Q$, there exists $b = \neg a \in Q$ such that $a \wedge b = 0$, i.e., Q is maximal.

Suppose that Q is maximal. We prove that $Q \in X_m$. From 2.2 it follows that $(Q, Q) \in R_\neg$. Moreover, if $Q \subseteq D \in X(\boldsymbol{A})$ and $(Q, D) \in R_\neg$, then $Q = D$, because Q is maximal. Thus, $(Q, Q) \in R_m$ and consequently $Q \in X_m$.

Suppose that $X_m = \max X(\boldsymbol{A})$. Let $Q \in X_m$. Then, $g(Q) = Q \in X_m$ because $\boldsymbol{A} \in \mathcal{SDMA}$.

Suppose that there exists $a \in A$ such that $\neg a \wedge a \neq 0$. Then there exists $P \in X(\boldsymbol{A})$ such that $\neg a \in P$ and $a \in P$. Let $Q \in \max X(\boldsymbol{A})$ such that $P \subseteq Q$. Since $Q \in X_m = \max X(\boldsymbol{A})$, $(Q, Q) \in R_m$. Since $\neg a \in P \subseteq Q$, $a \notin Q$, a contradiction. Thus, $\neg a \wedge a = 0$. □

It is known that a bounded distributive pseudocomplement lattice $\boldsymbol{A}$ is a Stone algebra if and only if for each prime filter P there exists at most an maximal filter U such that $P \subseteq U$ (see [2]). For weak-Stone algebras we can give the following result.

Theorem 2.5. *Let $\boldsymbol{A} \in \mathcal{SDMA}$. Then the following conditions are equivalent*

1. *The relation $R_\neg$ is euclidean, i.e., $R_\neg^{-1} \circ R_\neg \subseteq R_\neg$.*
2. *$\boldsymbol{A} \in \mathcal{WS}$.*
3. *$\boldsymbol{A} \in \mathcal{DMPL}$, and for any $P \in X(\boldsymbol{A})$ there exists a unique $Q \in X_m$ such that $(P, Q) \in R_m$.*

Proof. $(1 \Rightarrow 2)$ Assume that $R_\neg$ is euclidean and suppose that there exists $a \in A$ such that $\neg a \vee \neg\neg a \neq 1$. Then there exists $P, Q, D \in X(\boldsymbol{A})$ such that

$$\neg a, \neg\neg a \notin P, (P, Q) \in R_\neg, a \in Q, (P, D) \in R_\neg \text{ and } \neg a \in D.$$

Since $R_\neg$ is euclidean, $(D, Q) \in R_\neg$. Since $a \in Q$, we have $\neg a \notin D$, a contradiction.

$(2 \Rightarrow 3)$ As $\neg a \vee \neg\neg a = 1$, for any $a \in A$, we have

$$\begin{aligned}\neg 1 = 0 &= \neg(\neg a \vee \neg\neg a)\\ &= \neg\neg a \wedge \neg\neg\neg a = \neg\neg a \wedge \neg a.\end{aligned}$$

Thus, $\boldsymbol{A} \in \mathcal{DMPL}$.

Let $P \in X(\boldsymbol{A})$, and suppose that there exists $Q_1, Q_2 \in X_m$ such that

$$(P, Q_1) \in R_m \text{ and } (P, Q_2) \in R_m.$$

If $Q_1 \subsetneq Q_2$, there exists $a \in A$ such that $a \in Q_1$ and $a \notin Q_2$. So, $\neg a \notin P$, and consequently $\neg\neg a \in P$. It follows that $\neg a \notin Q_2$ and $\neg\neg a \in Q_2$. As $Q_2 \in X_m$, from (2) of Theorem 2.3, we have that $Q_2 = g^2(Q_2)$. So $a \in Q_2$, which is a contradiction. Thus $Q_1 \subseteq Q_2$. Similarly we can prove that $Q_2 \subseteq Q_1$.

$(3 \Rightarrow 1)$ Let $P, Z, D \in X(\boldsymbol{A})$ be such that $(P, Z) \in R_\neg$ and $(P, D) \in R_\neg$. By hypothesis, there exists a unique $Q \in X_m$ such that $(P, Q) \in R_m$, so $Z \subseteq Q$ and $D \subseteq Q$. Suppose that $(D, Z) \notin R_\neg$. Then there exists $a \in A$ such that $\neg a \in D$ and $a \in Z$. Thus, $\neg a, a \in Q$. Since $Q \in X_m$, by (2) of Theorem 2.3, $Q = g^2(Q)$, and thus $\neg\neg a \in Q$. Consequently,

$$\neg a \wedge \neg\neg a = 0 \in Q,$$

a contradiction. Thus, $(D, Z) \in R_\neg$, i.e., $R_\neg$ is euclidean. □

Let $\boldsymbol{A}$ be a bounded distributive lattice. The *Priestley space* [8] of $\boldsymbol{A}$ is the set $X(\boldsymbol{A})$, partially ordered by set inclusion, with the topology τ generated by the collection of all $\sigma(a)$, $\sigma(a)^c$. It is well known that $X(\boldsymbol{A})$

is compact, Hausdorff, and has as basis of clopen sets, namely the family $\sigma(\boldsymbol{A}) = \{\sigma(a) : a \in A\}$. The key fact is that the map σ is an isomorphism from $\boldsymbol{A}$ onto the lattice of clopen increasing subsets of $X(\boldsymbol{A})$.

A Priestley duality for $\neg$-lattices is given in [1] in terms of the $\neg$-spaces. Let us recall that a $\neg$-space is a pair $\mathcal{X} = \langle X, R\rangle$ such that:

(1) X is a Priestley space;
(2) R is a binary relation defined on X such that for each $x \in X$, $R(x)$ is a closed and decreasing subset of X;
(3) $\neg_R(U) = \{x \in X : R(x) \cap U = \varnothing\}$ is a clopen increasing for each clopen increasing U.

Let $\mathcal{X}$ be a $\neg$-space. Consider the relation R_m defined by $R_m(x) = \max R(x)$, for each $x \in X$. Consider the set $X_m = \bigcup\{\max R(x) : x \in X\}$.

Definition 2.6. *Let $\langle X, R\rangle$ be a $\neg$-space. We say that*

(1) $\mathcal{X}$ *is a* semi-De Morgan space *if the relation R_m is serial, and for every $x, y, z \in X_m$, if $(x, y) \in R_m$ and $(y, z) \in R_m$, then $x = y$.*
(2) $\mathcal{X}$ *is a* demi-p-space *if it is a semi-De Morgan space, and $(x, x) \in R_m$ for any $x \in X_m$.*
(3) $\mathcal{X}$ *is an* almost p-space *if it is a demi-p-space and $X = \max X$.*
(4) $\mathcal{X}$ *is a* weak Stone space *if it is a demi-p-space and the relation R is euclidean.*

By Theorem 2.3, Theorem 2.4, Theorem 2.5, and the results given in [1] for $\neg$-spaces we formulate the following theorem.

Theorem 2.7. *The $\neg$-spaces of semi-De Morgan algebras, demi-p-lattices, almost p-lattices, and weak Stone algebras are semi-De Morgan spaces, demi-p-spaces, almost p-spaces and weak Stone spaces, respectively.*

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