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THE TRANSITIVE AND CO–TRANSITIVE BLOCKING SETS IN $\mathbf{P}^2(\mathbb{F}_q)$

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Dedicated to the centenary of the birth of Ferenc Kárteszi (1907–1989).

Abstract. We classify the transitive and co-transitive blocking sets in a finite Desarguesian plane.

1. Introduction

A classical topic in finite geometries is the investigation of a projective space $\mathcal{P}$ admitting a collineation group G which preserves some special configuration $\mathcal{C}$ of $\mathcal{P}$, under suitable assumptions about $\mathcal{C}$ and the group–theoretical structure and action of G on $\mathcal{C}$.

Here we are interested in blocking sets of the finite Desarguesian projective plane $\mathbf{P}^2(\mathbb{F}_q)$.

A *blocking set* in a projective plane is a set of points which intersects every line; therefore, the smallest blocking sets are just the lines. Blocking sets containing a line will be called *trivial*. A point P of a blocking set B is called *essential* if $B \setminus \{P\}$ is not a blocking set. A blocking set B is said to be *minimal* when no proper subset of it is a blocking set. In particular, if B is a minimal blocking set then [5, Cor. 13.12, Th. 13.13]:

$$q + \sqrt{q} + 1 \leq |B| \leq q\sqrt{q} + 1.$$

The lower and upper bounds are attained by a Baer subplane and a Hermitian curve in a projective plane of square order, respectively. For other related results, an excellent source is [5, Ch. 13].

Suppose that B is a blocking set in $\mathbf{P}^2(\mathbb{F}_q)$ with automorphism group $\overline{G}_0 \leq \mathrm{P\Gamma L}(3,q)$. Then B is said to be *transitive* if $\overline{G}_0$ acts transitively on B, and *co–transitive* if $\overline{G}_0$ acts transitively on $\mathbf{P}^2(\mathbb{F}_q) \setminus B$.

The aim here is to classify transitive and co-transitive blocking sets in a Desarguesian plane of order q. We shall prove the following theorem.

Theorem 1.1. *Let B be a transitive and co–transitive blocking set of $\mathbf{P}^2(\mathbb{F}_q)$. Then one of the following occurs:*

(1) *B is a Baer subplane of $\mathbf{P}^2(\mathbb{F}_{q^2})$;*

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(2) *B is a Hermitian curve of* $\mathbf{P}^2(\mathbb{F}_{q^2})$*;*
(3) *B possibly is a union of Singer orbits in* $\mathbf{P}^2(\mathbb{F}_q)$ *and* $\overline{G}_0 \leq \Gamma\mathrm{L}(1, p^d) \leq \mathrm{GL}(d, p)$.

Remark 1.2. We point out that (3) in the theorem does not occur when $q = p^h$, with h odd. Moreover, since $q^2 + q + 1$ is always odd, it is clear that, when (3) occurs, B or its complement has even length, and hence an overgroup H of the normalizer of a subgroup of a Singer cyclic group must have even order. This means that H must be a subgroup of $\mathrm{P\Gamma L}(3, q)$. With the aid of MAGMA [3] we found a transitive and co–transitive blocking set of $\mathbf{P}^2(16)$ of size 91 admitting a group of order $2^2 \cdot 3 \cdot 7 \cdot 13$ that is the union of 13 subplanes of order 2, and a transitive and co-transitive blocking set of $\mathbf{P}^2(64)$ of size 1387 admitting a group of order $2 \cdot 3^3 \cdot 19 \cdot 73$ that is the union of 19 Baer subplanes. In both cases the blocking set is union of orbits of a subgroup of a Singer cyclic group.

Our main tool is the paper by Liebeck [6] where finite primitive affine permutation groups of rank three are classified, namely:

Theorem 1.3. *Let G be a finite primitive affine permutation group of rank three and degree $n = p^d$, with socle V, where $V \simeq (\mathbf{Z}_p)^d$ for some prime p, and let G_0 be the stabilizer of the zero vector in V. Then G_0 belongs to one of the following families:*
(1) 11 *infinite classes,* $A_1, \ldots, A_{11}$*;*
(2) *Extraspecial classes with* $G_0 \leq N_{\Gamma\mathrm{L}(d,p)}(R)$*, where R is a 2–group or a 3–group irreducible on V;*
(3) *Exceptional classes. In this case the socle L of $G_0/Z(G_0)$ is simple.*

For some results on blocking sets with special group actions, see [1, 2].

2. Some preliminaries

Let B be a blocking set of $\mathbf{P}^2(\mathbb{F}_q)$ such that there exists a subgroup $\overline{G}_0 \leq \mathrm{P\Gamma L}(3, q)$ acting transitively both on B and on its complement. Then $\overline{G}_0$ corresponds to a subgroup G_0 of $\mathrm{GL}(d, p)$ that has three orbits on vectors of $V(d, p)$, where p is a prime and $p^d = q^3$. Further, G_0 contains matrices corresponding to scalar multiplication by non–zero elements of $\mathbb{F}_q$. Of course, G_0 can be embedded in $\Gamma\mathrm{L}(3, q)$ and, as Liebeck notes, G_0 is embedded in $\Gamma\mathrm{L}(a, p^{d/a})$, with a minimal. Consequently, $a \leq 3$ and $q \leq p^{d/a}$.

First of all, we need the following crucial lemma.

Lemma 2.1. *Suppose that B is a non–trivial transitive and co–transitive blocking set of $\mathbf{P}^2(\mathbb{F}_q)$ with $\overline{G}_0 \leq \mathrm{P\Gamma L}(3, q)$ acting transitively on B and $\mathbf{P}^2(\mathbb{F}_q) \setminus B$. Let G_0 be the preimage of $\overline{G}_0$ in $\mathrm{GL}(d, p)$ and $H = V(d, p) \cdot G_0$. Then H is primitive on V.*

Proof. Assume that H is imprimitive on V. Let Ω be a block containing 0. It turns out that the two orbits of G_0 on non–zero vectors are $\Omega \setminus \{0\}$

and $V \setminus \Omega$. Let u, v be two vectors in Ω. Then $\Omega + v$ is a block containing $0 + v$ and $u + v$. Hence $\Omega + v = \Omega$. It follows that Ω is an $\mathbb{F}_p$–subspace of V. Since G_0 contains non–zero scalars in $\mathbb{F}_q$, Ω actually is an $\mathbb{F}_q$–subspace. In $\mathbf{P}^2(\mathbb{F}_q)$, the block Ω corresponds to a line, which is a trivial blocking set, and its complement cannot be a blocking set as well since it has an external line. □

All possibilities for G_0 and its two orbits on non–zero vector of $V(d,p)$ are provided by Liebeck's theorem [6, Theorem 2].

3. The case by case analysis

In this Section we begin with the case by case analysis. In many instances we need to look at the structure of the orbits and use known results on blocking sets.

3.1. **The class A_1.** Here G_0 is a subgroup of $GL(1,p^d)$ and contains $\mathbb{F}_q{}^*$. Such a subgroup is generated by ω^N and $\omega^e\alpha^s$, for certain N, e, s, where ω is a primitive element of $\mathbb{F}_{p^d}$ and α is a Frobenius automorphism of $\mathbb{F}_{p^d}$. If we write $p^d = q^a$, then N is a divisor of $(q^a - 1)/(q - 1)$. Let H_0 be the subgroup of $\Gamma\mathrm{L}(1,p^d)$ generated by ω^N. Then H_0 is a Singer cyclic subgroup of $\mathrm{GL}(1,p^d)$ and its orbits on $\mathbf{P}^2(\mathbb{F}_q)$ are called *Singer orbits*. Clearly, if G_0 has two orbits on $\mathbf{P}^2(\mathbb{F}_q)$, then each orbit is a union of Singer orbits. The two orbits of G_0 on non–zero vectors of $V(d,p)$ are described in [4]. The lengths of the orbits are $m_1(p^d - 1)/N$ and $(v - 1)m_1(p^d - 1)/N$, for some non–negative integers m_1, N, v, satisfying suitable arithmetic conditions.

3.2. **The class A_2.** In this case G_0 preserves a direct sum decomposition $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of $V(d,p)$ and have the same dimension. Here, one orbit is $(V_1 \cup V_2) \setminus \{0\}$ and the other is $\{v_1 + v_2 : 0 \neq v_1 \in V_1, 0 \neq v_2 \in V_2\}$. For any $\lambda \in \mathbb{F}_q{}^* \leq G_0$, $\lambda V_1 = V_1$ or V_2. Let $F = \{\lambda \in \mathbb{F}_q{}^* : \lambda V_1 = V_1\} \cup \{0\}$. It turns out that F is a subfield of $\mathbb{F}_q$ with order greater than $q/2$ and so must be $\mathbb{F}_q$. Thus V_1 and V_2 are subspaces of $V(3,q)$ of the same dimension, which is impossible.

3.3. **The class A_3.** Now G_0 preserves a tensor product decomposition $V = V_1 \otimes V_2$ over $\mathbb{F}_q$, where V_1 has dimension two over $\mathbb{F}_q$ and so it does not concern us.

3.4. **The class A_4.** In this case G_0 contains the group $\mathrm{SL}(a,s)$ as a normal subgroup, with $p^d = s^{2a}$. Here $q = s^2$, $a = 3$ and $p^d = q^3$ with $\mathrm{SL}(3,s)$ embedded in $\mathrm{GL}(d,p)$ as a subgroup of $\mathrm{SL}(a,s)$. Let e_1, e_2, e_3 be a basis for V over $\mathbb{F}_q$. Then, with respect to this basis, $\mathrm{SL}(a,s)$ consists of matrices of $\mathrm{SL}(a,q)$ with entries in the field with s elements. If G_0 has two orbits on non–zero vectors of V, then the orbits must be $\mathcal{O}_1 = \{\gamma \sum \lambda_i e_i : 0 \neq \gamma \in \mathbb{F}_q\}$ with the λ_i's in the field with s elements, not <u>all</u> zero, and $\mathcal{O}_2$ is the set of non–zero remaining vectors. In other words, $\overline{G}_0$ fixes a Baer subplane of $\mathbf{P}^2(\mathbb{F}_q)$.

3.5. **The class A_5.** Now the group G_0 contains as a normal subgroup the group $\mathrm{SL}(2,s)$, $p^d = s^6$. Here $q = s^3$ and $p^d = q^2$ with $\mathrm{SL}(2,s)$ embedded in $\mathrm{GL}(d,p)$ as a subgroup of $\mathrm{SL}(2,q)$; therefore, we can omit it.

3.6. **The class A_6.** In this case G_0 contains as a normal subgroup the group $\mathrm{SU}(a,q')$ and $p^d = ((q')^2)^a$, with $q = (q')^2$ and $a = 3$. Here, one orbit consists of the non–zero isotropic vectors of a certain non–degenerate Hermitian form, and the other orbit consists of the set of non–isotropic vectors with respect to the same form. Thus, the former orbit is a Hermitian curve of $\mathbf{P}^2(\mathbb{F}_q)$ which is a minimal blocking set of size $q\sqrt{q}+1$.

3.7. **The classes A_7–A_{11}.** These classes involve group representations acting on geometries with high dimension and so are of no importance to us.

3.8. **The Extraspecial classes.** In many cases here $G_0 \leq M$, where M is the normalizer in $\Gamma\mathrm{L}(a,q)$ of a 2–group R, where $p^d = q^a$ and $a = 2^m$, $m \geq 1$; either R is an extraspecial group 2^{1+2m} or R is isomorphic to $\mathbf{Z}_4 \circ 2^{1+2m}$. In any case p is odd.

There exist two types of extraspecial group 2^{1+2m} denoted by R_1^m and R_2^m; the former has the structure $D_8 \circ D_8 \circ \cdots \circ D_8$ (m copies) and the latter has the structure $D_8 \circ D_8 \circ \cdots \circ D_8 \circ Q_8$ ($m-1$ copies of D_8), where D_8 and Q_8 are the dihedral group and the quaternion group of order 8, respectively, and the "$\circ$" denotes a central product. The group $\mathbf{Z}_4 \circ 2^{1+2m}$ is a central product $\mathbf{Z}_4 \circ D_8 \circ D_8 \circ \cdots \circ D_8$ (m copies of D_8) and is denoted by R_3^m. Notice that R modulo its centre is an elementary abelian 2–group, that is, a $2m$–dimensional vector space over the field with two elements.

Just in one case $G_0 \leq M$, with M the normalizer in $\Gamma\mathrm{L}(3,4)$ of a 3–group of order 27. In this case, the non–trivial orbits on $V(3,4)$ have sizes 27 and 36. Thus, projectively, they have sizes 9 and 12. In $\mathbf{P}^2(\mathbb{F}_4)$ there exist only two blocking sets of size 9: the Hermitian curve, and another one as indicated in [5, Th. 13.23 ii) (b)]. The second blocking set is not minimal and admits an automorphism group of order 4.

3.9. **The Exceptional classes.** In these cases the socle L of $G_0/Z(G_0)$ is a simple group. There are thirteen possibilities for L, although in some instances to L there correspond different groups G_0. For example, if $L = Alt_5$, then there are seven possibilities for G_0. However, in these case, G_0 acts on vector spaces of dimension less then two. There is just one remaining case: $L = Alt_6$ with $(d,p) = (6,2)$ and L has an embedding in $\mathrm{PSL}(3,4)$. Here G_0 has an orbit of length $6 < 7$ which is a hyperoval $\mathcal{O}$ and, of course, $\mathcal{O}$ cannot be a blocking set.

Theorem 1 is completely proved.

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