Contributions to Discrete Mathematics

Volume 2, Number 2, Pages 185–204
ISSN 1715-0868

ASPECTS OF SINGULAR COFINALITY

ASSAF RINOT

Abstract. We study properties of closure operators of singular cofinality, and introduce several ZFC sufficient and equivalent conditions for the existence of antichain sequences in posets of singular cofinality. We also notice that the Proper Forcing Axiom implies the Milner-Sauer conjecture.

1. Introduction

1.1. **Background.** Assume $\langle P, \leq\rangle$ is a poset. We say that $C \subseteq P$ is *cofinal* in P iff $P = \underline{C}$, where $\underline{C} := \{x \in P \mid \exists y \in C(x \leq y)\}$. Define the *cofinality* of $\langle P, \leq\rangle$ to be $\mathrm{cf}(P) = \mathrm{cf}(P, \leq) := \min\{|C| \mid C$ is cofinal in $P\}$. For $x, y \in P$, we say that x and y are *incomparable* iff $x \not\leq y$ and $y \not\leq x$. $A \subseteq P$ is said to be an *antichain* iff x, y are **incomparable** for all distinct $x, y \in A$.

In [10], Pouzet proved his celebrated theorem stating that any updirected poset with no infinite antichain contains a cofinal subset which is isomorphic to a product of finitely many regular cardinals (for a proof, see, for instance, §4.13, and in particular §4.13.5, of [2]). Since any poset with no infinite antichain is the union of finitely many updirected subposets, we have:

Theorem 1.1 ([10])**.** *Assume $\langle P, \leq\rangle$ is a poset. If $\mathrm{cf}(P, \leq)$ is a singular cardinal, then P contains an infinite antichain.*

This lead to the formulation of a very natural conjecture, first appearing implicitly in [10], and then explicitly in a paper by Milner and Sauer.

Conjecture ([9])**.** *Assume $\langle P, \leq\rangle$ is a poset. If $\mathrm{cf}(P, \leq) = \lambda > \mathrm{cf}(\lambda) = \kappa$, then P contains an antichain of size κ.*

For $\mathcal{C}$, a class of posets, denote by $\mathrm{MS}(\mathcal{C})$ the statement: for all $\langle P, \leq\rangle \in \mathcal{C}$, P contains an antichain of size $\mathrm{cf}(\mathrm{cf}(P))$. For a cardinal λ, denote by MS_λ the statement $\mathrm{MS}(\{\langle P, \leq\rangle \mid \langle P, \leq\rangle$ is a poset of cofinality $\lambda\})$.

Thus, the Milner-Sauer conjecture is the statement $\forall\lambda(\lambda > \mathrm{cf}(\lambda) \to \mathrm{MS}_\lambda)$, and Suslin's hypothesis is the statement $\mathrm{MS}(\mathcal{T})$, where $\mathcal{T}$ denotes the class of all ever-branching $\aleph_1$-trees.

For a set A and a cardinal μ, let $[A]^{<\mu} := \{X \subseteq A \mid |A| < \mu\}$, $[A]^{\leq\mu} := \{X \subseteq A \mid |A| \leq \mu\}$, and $[A]^{\mu} := \{X \subseteq A \mid |A| = \mu\}$.

Received by the editors February 13, 2006, and in revised form March 26, 2007.
2000 *Mathematics Subject Classification.* Primary 03E04; Secondary 06A07.
Key words and phrases. Poset, antichain, singular cofinality.

The current state of the conjecture is the following:

Theorem 1.2 ([6],[12])**.** *Let λ be a singular cardinal. If* $\mathrm{cf}([\lambda]^{<\mathrm{cf}(\lambda)}, \subseteq) = \lambda$, *then* MS_λ.

This result was obtained independently by Magidor and by the author, using completely different arguments.

1.2. **Supplemental axioms for set theory.** Our paper is dedicated to the research of the combinatorial aspects of singular cofinality, and is carried out purely within ZFC, the usual axioms of set theory. However, the hypothesis "$\mathrm{cf}([\lambda]^{<\lambda}, \subseteq) = \lambda$", for a singular cardinal λ, is independent of ZFC.

Naturally, assuming supplemental axioms for set theory, more can be said on the Milner-Sauer conjecture, thus, in this small subsection, we briefly discuss results obtained in this direction.

The most famous supplemental axiom for set theory is probably the Generalized Continuum Hypothesis (GCH) stating that $2^\lambda = \lambda^+$ for any infinite cardinal λ. A weakening of the GCH is Shelah's Strong Hypothesis (SSH) from [14]. Fix a singular cardinal λ, and let $\kappa := \mathrm{cf}(\lambda)$. To define the SSH, let us say that $\langle \lambda, \mathbf{a}, \mathcal{F}\rangle$ is an appropriate triplet iff $\mathbf{a} \subseteq \lambda$ is a set of κ many regular cardinals satisfying $\sup(\mathbf{a}) = \lambda$, and $\mathcal{F}$ is an ultrafilter over $\mathbf{a}$ containing no bounded subsets. Consider the ultraproduct $\prod \mathbf{a}/\mathcal{F}$. It is a linearly ordered set and $\mathrm{cf}(\prod \mathbf{a}/\mathcal{F}) \geq \lambda^+$. Finally, define the *pseudopower*

$$\mathrm{pp}(\lambda) = \sup\left\{\mathrm{cf}\left(\prod \mathbf{a}/\mathcal{F}\right) \mid \langle \lambda, \mathbf{a}, \mathcal{F}\rangle \text{ is an appropriate triplet}\right\}.$$

The SSH states that $\mathrm{pp}(\lambda)$ is just λ^+ for any singular cardinal λ.

It is well known that the hypothesis of Theorem 1.2 is a consequence of the SSH (for a proof, see [12]). Recently, Viale proved in [15] that the SSH is a consequence of the Proper Forcing Axiom (PFA), and hence:

Corollary 1.3. PFA *implies the Milner-Sauer conjecture.*

For completeness, we mention that a poset $\langle P, \leq\rangle$ is *proper* iff for any regular uncountable cardinal μ, $\langle P, \leq\rangle$ preserves stationary subsets of $[\mu]^\omega$. Thus, PFA is the assertion that for any proper poset $\langle P, \leq\rangle$ and all sequences $\langle D_\alpha \mid \alpha < \omega_1\rangle$ of cofinal subsets of P, there is an updirected $G \subseteq P$ such that $\underline{G} \cap D_\alpha \neq \varnothing$ for all $\alpha < \omega_1$ (see further [1]). It is also worth mentioning that Suslin's hypothesis is, as well, a consequence of PFA.

We conclude this subsection with the following two additional results.

Theorem 1.4 ([3])**.** *Assuming* GCH *and the existence of a cardinal θ being $\theta^{+\omega_1+1}$-strong, it is possible to obtain a model of* ZFC *with* $\mathrm{cf}([\lambda]^{<\mathrm{cf}(\lambda)}, \subseteq) > \lambda > \mathrm{cf}(\lambda) + \mathrm{MS}_\lambda$

Theorem 1.5. *Let λ be a singular cardinal. If* $\mathrm{cf}([\lambda]^{<\mathrm{cf}(\lambda)}, \subseteq) = \lambda$, *then any poset of cofinality λ contains an antichain sequence of size λ and length* $\mathrm{cf}(\lambda)$.

Thus, Theorem 1.4 shows that Theorem 1.2 cannot be improved to an "iff" theorem, and Theorem 1.5 shows that the hypothesis of Theorem 1.2 will not only imply the existence of an antichain, but also the existence of an antichain sequence (see Definition 3.2).

The proof of Theorem 1.5 involves an inner-model argument combining Corollary 2.3 from [12] together with Theorem 1.2 from [8]. We shall not include the proof, but only mention that it is a straight-forward modification to the proof of Theorem 2.8 from [12].[1]

1.3. **Organization of this paper.** In Section 2, we study the abstract notion of cofinality in the general context of closure operators and derive basic properties of singular cofinality, most of them were already known in the private case of partial orders. One advantage of this approach is that it allows us to study posets of singular cofinality which are embedded in other posets, not necessarily of the same cofinality.

We also introduce the notion of *exact spectrum of cofinalities* which is being used extensively throughout this paper, and conclude this section, revealing a restriction on the spectrum induced by a potential counter-example.

In Section 3, we present a linear hierarchy of sequences (*normal, one-sided, upwards-extendible*) approximating Hajnal-Sauer's original definition of *antichain sequences* from [5], and prove that the existence of an upwards-extendible sequence is equivalent to the existence of an antichain sequence.

We also establish that in the context of the Milner-Sauer conjecture, it suffices to restrict our research to explore properties of *externally homogeneous* posets and principal one-sided sequences.

In Section 4, we prove MS($\mathcal{C}$) for several classes $\mathcal{C}$. We show that normal sequences with untight mutual relations between their components can be refined into antichain sequences, and also notice that a counter-example to the Milner-Sauer conjecture cannot embed a tree (or even a pseudotree) of the same cofinality.

The section is concluded with the formulation of an equivalent condition for the existence of antichain sequences in terms of *boundness by ideals.*

2. Closure operators

Definition 2.1. *Assume X is a set and* $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$ *is a function.* φ *is a* closure operator over X *iff*

(a) $A \subseteq B \subseteq X \Longrightarrow A \subseteq \varphi(A) \subseteq \varphi(B) \subseteq X$ *and*
(b) *for all* $A \subseteq X$, $\varphi(\varphi(A)) = \varphi(A)$.

φ *is a* topological closure operator over X *if, additionally*

(c) $\varphi(\varnothing) = \varnothing$ *and*
(d) $A, B \subseteq X \Longrightarrow \varphi(A) \cup \varphi(B) = \varphi(A \cup B)$.

Note that by property (a), $A, B \subseteq X$ *implies* $\varphi(A) \cup \varphi(B) \subseteq \varphi(A \cup B)$.

[1]Added in proof: a major improvement to both Theorems 1.4 and 1.5 has recently been established. See [13].

Closure operators are common objects in mathematics. Except the obvious example of topological closure, the operator $A \mapsto \mathrm{Span}(A)$ for a subset A of a vector space V is a closure operator, as well. The operation $A \mapsto \langle A\rangle$ (the subgroup generated by A) for a subset A of a group G is a closure operator. Another important example is described in the following.

Definition 2.2. *Assume $\langle P, \leq\rangle$ is a binary structure, and fix $A \subseteq P$.*
The downward closure *of A, is* $\underline{A} := \{x \in P \mid \exists y \in A(x \leq y)\}$.
The upward closure *of A, is* $\overline{A} := \{x \in P \mid \exists y \in A(y \leq x)\}$.

Notice that if $\langle P, \leq\rangle$ is reflexive and transitive, then the map $A \mapsto \underline{A}$ (for all $A \subseteq P$) defines a topological closure operator over P.

We now define the analogue of dimension.

Definition 2.3. *Suppose φ is a closure operator over some set P.*
For a subset $A \subseteq P$, denote $\mathrm{cf}_\varphi(A) := \min\{|B| \mid B \subseteq P, A \subseteq \varphi(B)\}$.
The spectrum of cofinalities of φ *is* $\mathrm{Spec}(\varphi) := \{\mathrm{cf}_\varphi(A) \mid A \in [P]^{<\mathrm{cf}_\varphi(P)}\}$.
The exact spectrum *is* $\mathrm{ESpec}(\varphi) := \{\mathrm{cf}_\varphi(A) \mid A \subseteq P, |A| = \mathrm{cf}_\varphi(A)\}$.
For a cardinal $\mu \leq \mathrm{cf}_\varphi(P)$, let $\mathrm{ESpec}_\mu(\varphi) := \{A \subseteq P \mid |A| = \mathrm{cf}_\varphi(A) = \mu\}$.

Lemma 2.4. *Suppose φ is a closure operator over a set P, then*
(a) *for $A \subseteq P$, $\mathrm{cf}_\varphi(\varphi(A)) = \mathrm{cf}_\varphi(A) \leq |A|$;*
(b) *for $A \subseteq B \subseteq P$, $\mathrm{cf}_\varphi(A) \leq \mathrm{cf}_\varphi(B)$; and*
(c) *for an indexed family $\langle A_i \subseteq P \mid i \in I\rangle$,*
$$\mathrm{cf}_\varphi\left(\bigcup_{i\in I} A_i\right) \leq \sum_{i\in I} \mathrm{cf}_\varphi(A_i).$$

Proof. Easy. □

Corollary 2.5. $\mathrm{ESpec}(\varphi) = \mathrm{Spec}(\varphi) \cup \{\mathrm{cf}_\varphi(P)\}$. *In particular, if $\lambda = \mathrm{cf}_\varphi(P)$, then $\mathrm{ESpec}_\lambda(\varphi) \neq \varnothing$.*

Proof. ($\subseteq$) By definition, we can choose $A \in [P]^{\mathrm{cf}_\varphi(P)}$ such that $P \subseteq \varphi(A)$. By the preceding lemma, $\mathrm{cf}_\varphi(P) \leq \mathrm{cf}_\varphi(A) \leq |A| = \mathrm{cf}_\varphi(P)$, hence, A is a witness that $\mathrm{cf}_\varphi(P) \in \mathrm{ESpec}(\varphi)$. Since $\mathrm{ESpec}(\varphi) \setminus \{\mathrm{cf}_\varphi(P)\} \subseteq \mathrm{Spec}(\varphi)$, we conclude that $\mathrm{ESpec}(\varphi) \subseteq \mathrm{Spec}(\varphi) \cup \{\mathrm{cf}_\varphi(P)\}$.

The proof of the other inclusion ($\supseteq$) is similar. □

We now introduce an essential property of singular cofinality.

Lemma 2.6 ([5],[8],[4])**.** *Suppose φ is a closure operator over a set P, and $\lambda \in \mathrm{ESpec}(\varphi)$ is a singular cardinal, then*

(a) $\mathrm{Spec}_\varphi(P) \cap \lambda$ is unbounded in λ and
(b) $T := \{\mu \in \mathrm{ESpec}(\varphi) \mid \mathrm{cf}(\mu) = \mu < \lambda\}$ is unbounded in λ.

If $\mathrm{cf}(\lambda) > \aleph_0$, then, also
(c) $\mathrm{ESpec}(\varphi) \cap \lambda$ is a club in λ and
(d) $S := \{\mu \in \mathrm{ESpec}(\varphi) \mid \mathrm{cf}(\mu) < \mu < \lambda\}$ is stationary in λ.

Proof. (a) Pick $P' \in \mathrm{ESpec}_\lambda(\varphi)$ and let $\kappa := \mathrm{cf}(\lambda)$. By $\kappa < \lambda$, there exists a sequence $\langle A_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$ with $\bigcup_{\alpha<\kappa} A_\alpha = P'$. It follows that

$$\lambda = \mathrm{cf}_\varphi(P') = \mathrm{cf}_\varphi\left(\bigcup_{\alpha<\kappa} A_\alpha\right) \leq \sum_{\alpha<\kappa} \mathrm{cf}_\varphi(A_\alpha),$$

and by $\kappa = \mathrm{cf}(\lambda)$, we must conclude that $\sup\{\mathrm{cf}_\varphi(A_\alpha) \mid \alpha < \kappa\} = \lambda$.

(b) Put $\delta := \mathrm{otp}(\mathrm{ESpec}(\varphi))$ and let $\sigma : \delta \leftrightarrow \mathrm{ESpec}(\varphi)$ be the order-preserving bijection. For $\beta < \delta$, we have $\sup(\mathrm{Spec}_\varphi(P) \cap \sigma(\beta+1)) = \sigma(\beta)$, thus, by applying the previous item to $\sigma(\beta+1) \in \mathrm{ESpec}(\varphi)$, we must conclude that $\sigma(\beta+1)$ is regular.

(c) $\mathrm{ESpec}(\varphi)$ is clearly closed. To see unboundness, pick $\mu < \lambda$. By induction on $n < \omega$, use item (a) to pick $A_0 \in [P]^{<\lambda}$ with $\mathrm{cf}_\varphi(A_0) > \mu$, and $A_{n+1} \in [P]^{<\lambda}$ with $\mathrm{cf}_\varphi(A_{n+1}) > |A_n|$ for all $n < \omega$. Finally, put $A := \bigcup_{n<\omega} A_n$ and notice that $\mu < \mathrm{cf}_\varphi(A_0) < \mathrm{cf}_\varphi(A) = |A| < \lambda$.

(d) Because S contains the intersection of the club, $\mathrm{ESpec}(\varphi) \cap \lambda$, with the stationary set $S_1 := \{\mu < \lambda \mid \aleph_0 = \mathrm{cf}(\mu) < \mu\}$. □

Definition 2.7. *Suppose φ is a closure operator over a set P. For a cardinal μ, we define a* $\mathrm{cf}(\mu)$*-complete ideal*

$$\mathcal{I}_\mu(\varphi) := \{A \subseteq P \mid \mathrm{cf}_\varphi(A) < \mu\}.$$

Notice that always $[P]^{<\mu} \subseteq \mathcal{I}_\mu(\varphi)$ and $\mathrm{cf}(\mathcal{I}_\mu(\varphi), \subseteq) \leq \mathrm{cf}([P]^{<\mu}, \subseteq)$.

Lemma 2.8. *If φ is a closure operator over a set P, and $\mathrm{cf}_\varphi(P) = \lambda$ is a singular cardinal, then $\mathcal{I}_\mu(\varphi) \neq \mathcal{I}_\lambda(\varphi)$ for all $\mu < \lambda$.*

Proof. Assume the existence of $\mu < \lambda$ with $\mathcal{I}_\mu(\varphi) = \mathcal{I}_\lambda(\varphi)$, then in particular $[P]^{<\lambda} \subseteq \mathcal{I}_\mu(\varphi)$ and $\sup(\mathrm{Spec}(\varphi)) = \sup\{\mathrm{cf}_\varphi(A) \mid A \in [P]^{<\lambda}\} \leq \mu < \lambda$, contradicting Lemma 2.6. □

Thus, for instance, if $\langle X, O\rangle$ is a topological space and $d(X) = \lambda$ is a singular cardinal, then for any cardinal $\mu < \lambda$, there exists a subspace $Y \in [X]^{<\lambda}$ such that $d(Y) > \mu$. (Recall that $d(X) := \min\{|D| \mid D \subseteq X, \overline{D} = X\}$, e.g., $\langle X, O\rangle$ is separable if $d(X) = \aleph_0$.)

Definition 2.9 (folklore)**.** *Suppose $\mathcal{I}$ is a non-trivial proper ideal over a set P, that is $\{\{x\} \mid x \in P\} \subseteq \mathcal{I}$ and $P \notin \mathcal{I}$. Let $\mathrm{cov}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} = P\}$, $\mathrm{non}(\mathcal{I}) = \min\{|A| \mid A \subseteq P, A \notin \mathcal{I}\}$.*

It is easy to see that $\mathrm{cov}(\mathcal{I}) \leq \mathrm{cf}(\mathcal{I}, \subseteq)$. Note also that for a closure operator φ, an infinite cardinal $\mu \in \mathrm{ESpec}(\varphi)$ iff $\mathrm{non}(\mathcal{I}_\mu(\varphi)) = \mu$.

Lemma 2.10. *Assume $\mathcal{I}$ is a non-trivial proper ideal over a set P. Suppose $\mathcal{C} \subseteq \mathcal{I}$ and a cardinal λ satisfying $|\mathcal{C}| \leq \lambda \leq \mathrm{cov}(\mathcal{I})$. Then, there exists $X \in [P]^{\mathrm{cf}(\lambda)}$ such that $|A \cap X| < |X|$ for each $A \in \mathcal{C}$. In particular, $X \nsubseteq \bigcup \mathcal{B}$ for all $\mathcal{B} \in [\mathcal{C}]^{<\mathrm{cf}(\lambda)}$.*

Proof. Let $\{A_i \mid i < \lambda\}$ be an enumeration of $\mathcal{C}$. Put $\kappa := \mathrm{cf}(\lambda)$ and fix an increasing sequence of ordinals $\{\lambda_\alpha \mid \alpha < \kappa\}$ converging to λ. We build a kind of Luzin set $X = \{x_\alpha \mid \alpha < \kappa\}$ for $\mathcal{C}$, by induction on $\alpha < \kappa$.

Suppose $\{x_\beta \mid \beta < \alpha\}$ have already been defined and let $Y_\alpha := \{A_i \mid i < \lambda_\alpha\} \cup \{\{x_\beta\} \mid \beta < \alpha\}$. Since $Y_\alpha \in [\mathcal{I}]^{<\mathrm{cov}(\mathcal{I})}$, we have that $X \setminus \bigcup Y_\alpha \neq \varnothing$, so we may pick $x_\alpha \in X \setminus \bigcup Y_\alpha$. This completes the construction. □

Corollary 2.11. *Suppose $\mathcal{I}$ is a non-trivial proper ideal over a set P. If $\mathrm{cov}(\mathcal{I}) = \mathrm{cf}(\mathcal{I}, \subseteq)$, then $\mathrm{non}(\mathcal{I}) \leq \mathrm{cf}(\mathrm{cf}(\mathcal{I}, \subseteq))$.*

Proof. Pick $\mathcal{C} \in [\mathcal{I}]^{\mathrm{cf}(\mathcal{I},\subseteq)}$ such that for each $A \in \mathcal{I}$, there exists $B \in \mathcal{C}$ with $A \subseteq B$. Now, since $|\mathcal{C}| = \mathrm{cov}(\mathcal{I})$, we may appeal to the preceding lemma and find a Luzin set $X \in [P]^{\mathrm{cf}(\mathrm{cov}(\mathcal{I}))}$ for $\mathcal{C}$. Clearly, $|X| = \mathrm{cf}(\mathrm{cf}(\mathcal{I}, \subseteq))$.

Since $X \nsubseteq B$ for all $B \in \mathcal{C}$, we conclude that $X \notin \mathcal{I}$. □

It is possible to weaken the hypothesis of the preceding corollary, obtaining the following convergence theorem.

Lemma 2.12. *Assume $\mathcal{I}$ is a non-trivial proper ideal over a set P. Suppose also that θ, λ are cardinals, and for each $\alpha < \theta$, $\mathcal{I}_\alpha$ is a non-trivial proper ideal over P satisfying $\mathrm{cov}(\mathcal{I}_\alpha) \geq \lambda \geq \mathrm{cf}(\mathcal{I}, \subseteq)$.*

If $\mathcal{I} = \bigcup_{\alpha<\theta} \mathcal{I}_\alpha$, then $\mathrm{non}(\mathcal{I}) \leq \theta + \mathrm{cf}(\lambda)$.

Proof. Fix a cofinal subset $\mathcal{C} \subseteq \mathcal{I}$ with $|\mathcal{C}| \leq \lambda$. For $\alpha < \theta$, put $\mathcal{C}_\alpha := \mathcal{C} \cap \mathcal{I}_\alpha$. By $|\mathcal{C}_\alpha| \leq \lambda \leq \mathrm{cov}(\mathcal{I}_\alpha)$ and Lemma 2.10, we may pick $X_\alpha \in [P]^{\mathrm{cf}(\lambda)}$ such that $X_\alpha \nsubseteq A$ for all $A \in \mathcal{C}_\alpha$. Let $X := \bigcup_{\alpha<\theta} X_\alpha$. Then $|X| \leq \theta + \mathrm{cf}(\lambda)$. Finally, if $X \in \mathcal{I}$, then by the choice of $\mathcal{C}$, there exists some $A \in \mathcal{C}$ with $X \subseteq A$, and in particular, there exists some $\alpha < \theta$ with $A \in \mathcal{C}_\alpha$. It follows that $X_\alpha \nsubseteq A$, contradicting $X_\alpha \subseteq X \subseteq A$. □

Corollary 2.13 ([7])**.** *Assume φ is a closure operator over a set P, and $\mathrm{cf}_\varphi(P) = \lambda > \mathrm{cf}(\lambda) = \kappa$. If μ is a cardinal satisfying $\kappa < \mu \leq \lambda$, then $\mathrm{cf}(\mathcal{I}_\mu(\varphi), \subseteq) > \lambda$.*

Proof. For notational simplicity, put $\mathcal{I}_\mu := \mathcal{I}_\mu(\varphi)$. Assume first $\kappa < \mu < \lambda$. By $\mathrm{cf}_\varphi(P) = \lambda$ and Lemma 2.4.c it is clear that $\mathrm{cov}(\mathcal{I}_\mu) = \lambda$. It follows from Corollary 2.11 that if $\mathrm{cf}(\mathcal{I}_\mu, \subseteq) = \lambda$, then $\mathrm{non}(\mathcal{I}_\mu) \leq \kappa$, contradicting the fact that $[P]^\kappa \subseteq [P]^{<\mu} \subseteq \mathcal{I}_\mu$.

Consider now $\mathcal{I}_\lambda$. Let $\langle \lambda_\alpha \mid \alpha < \kappa \rangle$ be a strictly increasing sequence of cardinals converging to λ. Then $\mathcal{I}_\lambda = \bigcup_{\alpha<\kappa} \mathcal{I}_{\lambda_\alpha}$.

Finally, if $\mathrm{cf}(\mathcal{I}_\lambda, \subseteq) \leq \lambda$, then by Lemma 2.12, $\mathrm{non}(\mathcal{I}_\lambda) \leq \kappa$, contradicting the fact that $[P]^\kappa \subseteq \mathcal{I}_\lambda$. □

Thus, for instance, if $\langle G, \cdot, 1 \rangle$ is a group, and $|G| = \lambda$ is a singular cardinal, then G has more than λ many subgroups. To be more concrete, if $|G| = \aleph_\omega$, then the set $\{H < G \mid H \text{ is countable}\}$ is of cardinality $> \aleph_\omega$.

Corollary 2.14. *If φ is a closure operator over a set P, and $\mathrm{cf}_\varphi(P) = \lambda > \mathrm{cf}(\lambda) = \kappa$, then $\mathcal{I}_{\kappa^+}(\varphi) \neq \mathcal{I}_\omega(\varphi)$.*

It follows that if $\langle P, \lhd\rangle$ is a μ-updirected poset for some cardinal μ, then either $\mathrm{cf}\langle P, \lhd\rangle \leq 1$ or $\mathrm{cf}(\mathrm{cf}(P, \lhd)) \geq \mu$.

We shall now revisit Definition 2.3 for the special case of partial orders.

Definition 2.15. *Assume $\langle P, \leq\rangle$ is poset, and $A \subseteq P' \subseteq P$. Let $\mathrm{cf}_{P'}(A) := \min\{|B| \mid B \subseteq P', A \subseteq \underline{B}\}$, and $\mathrm{cf}(P') := \mathrm{cf}_{P'}(P')$.*

Let φ denote the corresponding (topological) closure operator $A \mapsto \underline{A}$ (for all $A \subseteq P$), then denote $\mathrm{Spec}(P) := \mathrm{Spec}(\varphi), \mathrm{ESpec}(P) := \mathrm{ESpec}(\varphi)$, and $\mathcal{I}_\mu(P) := \mathcal{I}_\mu(\varphi)$ for any cardinal μ.

Finally, for $P' \subseteq P$ and a cardinal $\mu \leq \mathrm{cf}_P(P')$, we define

$$\mathrm{ESpec}_\mu(P') := \{A \subseteq P' \mid \langle A, \leq\rangle \text{ is well-founded and } |A| = \mathrm{cf}_P(A) = \mu\}.$$

To help our reader get used to the definition, let us take a look at the sets $\Gamma_1 := \{\mathrm{cf}_P(A) \mid A \subseteq P\}$ and $\Gamma_2 := \{\mathrm{cf}(A) \mid A \subseteq P\}$ for some poset $\langle P, \leq\rangle$. By Corollary 2.5, we have $\Gamma_1 = \mathrm{ESpec}(P)$ and by Lemma 2.4, we know that Γ_1 is closed (i.e. $\bigcup \mathcal{A} \in \Gamma_1$ for all $\mathcal{A} \subseteq \Gamma_1$). Also, it is proved in [8] that $\Gamma_2 \supseteq \{\mu < \mathrm{cf}(P) \mid \mu = \mathrm{cf}(\mu)\}$.

Now, consider the case $\langle P, \leq\rangle = \langle \aleph_\omega, \in\rangle$. Since it is linearly ordered, we have $\Gamma_1 = \{0, 1, \mathrm{cf}(P)\}$. However, $\Gamma_2 = \{\mu < \aleph_\omega \mid \mu = \mathrm{cf}(\mu)\}$. In particular, Γ_2 is not closed, and $\Gamma_1 \not\supseteq \{\mu < \mathrm{cf}(P) \mid \mu = \mathrm{cf}(\mu)\}$.

The essence of being *exact* is the following.

Lemma 2.16. *Suppose $\langle P, \leq\rangle$ is a poset, and $A \in \mathrm{ESpec}_\mu(P)$ for some μ. Then, whenever $A \subseteq Q \subseteq P$, we have, $\mathrm{cf}(A) = \mathrm{cf}_Q(A) = \mathrm{cf}_P(A)$.*

Lemma 2.17 ([10])**.** *For a poset $\langle P, \leq\rangle$, $\mathrm{ESpec}(P) = \{\mu \mid \mathrm{ESpec}_\mu(P) \neq \varnothing\}$.*

Proof. Let φ be the corresponding operator. To see the non-trivial direction $(\subseteq)$, fix $\mu \in \mathrm{ESpec}(P)$. Pick $\{x_\alpha \mid \alpha < \mu\} \in \mathrm{ESpec}_\mu(\varphi)$, and let $A := \{x_\alpha \mid \forall \beta < \alpha (x_\beta \not\geq x_\alpha)\}$. Evidently, $A \in \mathrm{ESpec}_\mu(P)$, thus, $\mathrm{ESpec}_\mu(P) \neq \varnothing$. □

Lemma 2.18. *Assume $\langle P, \leq\rangle$ is a poset and $P' \subseteq P$ is a* cofinal subset *(i.e., $P \subseteq \underline{P'}$), then*

(a) *for $A \subseteq P'$, $\mathrm{cf}_{P'}(A) = \mathrm{cf}_P(A)$ and*
(b) *$\mathrm{cf}(\mathcal{I}_\mu(P), \subseteq) = \mathrm{cf}(\mathcal{I}_\mu(P'), \subseteq)$ for any cardinal μ.*

Proof. (a) Let $A \subseteq P'$. Clearly $\mathrm{cf}_P(A) \leq \mathrm{cf}_{P'}(A)$. To see the other inequality, pick $X \in P$ of minimal cardinality such that $A \subseteq \underline{X}$. For $\sigma := |X|$, fix an enumeration $X = \{x_\alpha \mid \alpha < \sigma\}$. By $P \subseteq \underline{P'}$, for all $\alpha < \sigma$ we may pick $y_\alpha \in P'$ such that $x_\alpha \leq y_\alpha$, then, $Y = \{y_\alpha \mid \alpha < \sigma\}$ witnesses $\mathrm{cf}_{P'}(A) \leq \mathrm{cf}_P(A)$.

(b) Recall that $\mathcal{I}_\mu(P') := \{A \subseteq P' \mid \mathrm{cf}_{P'}(A) < \mu\}$. Let $f : \mathcal{I}_\mu(P) \to [P']^{<\mu}$ be a function such that $X \subseteq \underline{f(X)}$ for all $X \in \mathcal{I}_\mu(P)$. To see that such a function exists, fix $X \in \mathcal{I}_\mu(P)$ and repeat the arguments of the preceding item: by definition, there exists $Y \in [P]^{<\mu}$ with $X \subseteq \underline{Y}$. Since $P \subseteq \underline{P'}$, for all $y \in Y$, we may find $y' \in P'$ with $y \leq y'$, so let $f(X) := \{y' \mid y \in Y\}$.

Put $\theta := \mathrm{cf}(\mathcal{I}_\mu(P), \subseteq)$ and $\theta' := \mathrm{cf}(\mathcal{I}_\mu(P'), \subseteq)$. Pick $\mathcal{C} \in [\mathcal{I}_\mu(P)]^\theta$ and $\mathcal{C}' \in [\mathcal{I}_\mu(P')]^{\theta'}$ witnessing the cofinalities.

To prove $\theta \leq \theta'$, we show that $\{\underline{f(B)} \mid B \in \mathcal{C}'\}$ is cofinal in $\mathcal{I}_\mu(P)$. Assume $A \in \mathcal{I}_\mu(P)$. By $f(A) \in [P']^{<\mu} \subseteq \mathcal{I}_\mu(P')$, there exists $B \in \mathcal{C}'$ such that $f(A) \subseteq B$. It follows that $A \subseteq \underline{f(A)} \subseteq \underline{B} \subseteq \underline{f(B)}$.

To prove $\theta' \leq \theta$, we show that $\{P' \cap \underline{f(B)} \mid B \in \mathcal{C}\}$ is cofinal in $\mathcal{I}_\mu(P')$. Assume $A \in \mathcal{I}_\mu(P')$. By $f(A) \in [P']^{<\mu} \subseteq \mathcal{I}_\mu(P)$, there exists $B \in \mathcal{C}$ such that $f(A) \subseteq B \cap P'$. By $A \subseteq \underline{f(A)} \subseteq \underline{B}$ and $B \subseteq \underline{f(B)}$, we conclude that $A \subseteq P' \cap \underline{f(B)}$. □

The proof of the following lemma can essentially be found in [7].

Lemma 2.19. *Assume $\langle P, \leq\rangle$ is a well-founded poset, and let κ denote the minimal cardinality such that P does not contain an antichain of size κ. For any ideal $\mathcal{I} \subseteq \mathcal{P}(P)$ satisfying $(A \in \mathcal{I} \to \underline{A} \in \mathcal{I})$, there exists a family $\mathcal{S} \subseteq [P]^{<\kappa}$ such that* $\mathrm{cf}(\mathcal{I}, \subseteq) \leq \mathrm{cf}(\mathcal{S}, \supseteq)$.

Proof. Consider the function $\mu : \mathcal{P}(P) \to [P]^{<\kappa}$ defined by letting $\mu(X) := \{x \in X \mid \forall y \in X(y \not< x)\}$ for all $X \subseteq P$. The definition is good because $\mu(X)$ is an antichain for all $X \subseteq P$. Denote $\phi(X) := \mu(P \setminus X)$ for all $X \subseteq P$.

Put $\mathcal{S} := \{\phi(X) \mid X \in \mathcal{I}\}$. We claim that $\mathrm{cf}(\mathcal{I}, \subseteq) \leq \mathrm{cf}(\mathcal{S}, \supseteq)$. To see this, fix $\mathcal{C}' \subseteq \mathcal{S}$ witnessing the value of $\mathrm{cf}(\mathcal{S}, \supseteq)$ and let $\mathcal{C} := \{P \setminus \overline{Y} \mid Y \in \mathcal{C}'\}$.

Fix $A \in \mathcal{I}$, we shall find $B \in \mathcal{C}$ with $A \subseteq B$. Put $A' := \underline{A}$. By the hypothesis, $A' \in \mathcal{I}$, and clearly $A \subseteq A'$. Since $A' = \underline{A'}$, we have $P \setminus A' = \overline{P \setminus A'}$. By well-foundedness, $P \setminus A' = \overline{\phi(A')}$. Find $B' \in \mathcal{C}'$ such that $B' \subseteq \phi(A')$, then $\overline{\phi(A')} \supseteq \overline{B'}$ and $A \subseteq A' = P \setminus \overline{\phi(A')} \subseteq P \setminus \overline{B'}$.

The same argument also shows that $\mathcal{C} \subseteq \mathcal{I}$. □

Corollary 2.20. *If there exists a poset $\langle P, \leq\rangle$, $\mathrm{cf}(P) = \lambda > \mathrm{cf}(\lambda)$ with no antichains of size κ, then there exists $\mathcal{S} \subseteq [\lambda]^{<\kappa}$ such that $\mathrm{cf}(\mathcal{S}, \supseteq) > \lambda$.*

Proof. Pick $P' \in \mathrm{ESpec}_\lambda(P)$ with $P \subseteq \underline{P'}$. By Lemma 2.18 and Corollary 2.13, $\mathrm{cf}(\mathcal{I}_\lambda(P'), \subseteq) > \lambda$. Thus, by the preceding lemma, we may find $\mathcal{S}' \in [P']^{<\kappa}$ with $\mathrm{cf}(\mathcal{S}', \supseteq) > \lambda$, and by $|P'| = \lambda$, this indicates the existence of $\mathcal{S} \in [\lambda]^{<\kappa}$ with $\mathrm{cf}(\mathcal{S}, \supseteq) > \lambda$. □

Corollary 2.21 ([10])**.** *If $\langle P, \leq\rangle$ is a poset of singular cofinality, then P contains an infinite antichain.*

Corollary 2.22. *Suppose $\langle P, \leq\rangle$ is a poset of singular cofinality λ, with no antichains of size* $\mathrm{cf}(\lambda)$. *If λ is minimal in that sense, then* $|\mathrm{ESpec}(P)| = \mathrm{cf}(\lambda)$.

Proof. By the preceding corollary, $\kappa := \mathrm{cf}(\lambda) > \aleph_0$. Recalling Lemma 2.6, we have that $C := \mathrm{ESpec}(P) \setminus \{\lambda\}$ is a club in λ, and hence $|C| \geq \mathrm{cf}\,|C| = \kappa$.

Assume towards a contradiction that $|C| > \kappa$. Then one can find a singular $\mu \in C$ such that $\mathrm{cf}(\mu) = \kappa$. Fix $\mu \in \mathrm{ESpec}(P)$ with $\lambda > \mu > \mathrm{cf}(\mu) = \kappa$. By minimality of λ, if $P' \in \mathrm{ESpec}_\mu(P)$, then P' (and hence P) contains an antichain of size κ, yielding a contradiction. □

3. Sequences of subposets

Definition 3.1. *Assume $\mathcal{A} = \langle A_j \mid j \in J\rangle$ is an indexed sequence, $I \subseteq J$. We write $X \subseteq^I \mathcal{A}$ iff $X = \mathrm{Im}(\phi)$ for some choice function $\phi \in \prod_{i\in I} A_i$.*

Our journey begins with the following definition due to Hajnal and Sauer.

Definition 3.2 ([5]). *Assume $\langle P, \leq\rangle$ is a poset, $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa\rangle$ is a family of sets, and $P' \subseteq P$. $\mathcal{A}$ is said to be an* antichain sequence for P' *iff*

(a) *for all $\beta < \alpha < \kappa$, $|A_\beta| \leq |A_\alpha|$ and $A_\alpha \subseteq P'$ and*
(b) *for all $I \subseteq \kappa$ and $X \subseteq^I \mathcal{A}$, X is an antichain.*

κ is considered to be the length *of the antichain sequence. The cardinality and cofinality (with respect to P) of $\bigcup_{\alpha<\kappa} A_\alpha$ will, respectively, be called the* size *and* cofinality *of the antichain sequence $\mathcal{A}$.*

The next lemma motivates the definition of an antichain sequence.

Lemma 3.3. *Assume $\langle P, \leq\rangle$ is a poset and $P' \in \mathrm{ESpec}_\lambda(P)$ for cardinals $\lambda > \mathrm{cf}(\lambda) = \kappa$. If there exists an antichain sequence for P' of length κ and size λ, then P' (and hence P) contains λ^κ antichains of size κ.*

Proof. Fix $\mathcal{A} := \langle A_\alpha \mid \alpha < \kappa\rangle$ like in the hypothesis. For all $\alpha < \kappa$, set $\lambda_\alpha = |A_\alpha|$. Finally, since $\langle \lambda_\alpha \mid \alpha < \kappa\rangle$ is non-decreasing, converging to λ:

$$\left|\left\{\mathrm{Im}(\phi) \mid \phi \in \prod_{\alpha<\kappa} A_\alpha\right\}\right| = \prod_{\alpha<\kappa} \lambda_\alpha = \lambda^\kappa.$$

□

We now start approximating Definition 3.2 in the following way.

Definition 3.4. *Assume $\langle P, \leq\rangle$ is a poset and $P' \in \mathrm{ESpec}_\lambda(P)$ for some singular cardinal λ. Put $\kappa := \mathrm{cf}(\lambda)$. A sequence $\mathcal{A} = \langle A_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$ is a* normal sequence for P' *iff it satisfies*

(a) *for all $\alpha < \kappa$, $\mathrm{cf}_P(A_\alpha) > (|V_\alpha| + \kappa)^+$, where $V_\alpha := \bigcup_{\beta<\alpha} A_\beta$ and*
(b) *$\sup\{\mathrm{cf}_P(A_\alpha) \mid \alpha < \kappa\} = \lambda$.*

It is a one-sided sequence for P' *if, additionally*

(c) *for each $\beta < \alpha < \kappa$, $\underline{A_\beta} \cap A_\alpha = \varnothing$.*

The length *of the sequence $\mathcal{A}$ is κ, and the* cofinality *of $\mathcal{A}$ is $\mathrm{cf}_P\left(\bigcup_{\alpha<\kappa} A_\alpha\right)$. Notice that the cofinality in items (a),(b) is also calculated according to P.*

Lemma 3.5. *Assume $\langle P, \leq\rangle$ is a poset and $P' \in \mathrm{ESpec}_\lambda(P)$ for some cardinal $\lambda > \mathrm{cf}(\lambda) = \kappa$. Then*

(a) *for any sequence of cardinals $\langle \lambda_\alpha \mid \alpha < \kappa\rangle$ cofinal in λ, there exists a normal sequence for P', $\mathcal{A} = \langle A_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$, such that $\mathrm{cf}_P(A_\alpha) > \lambda_\alpha$ for all $\alpha < \kappa$ and*
(b) *any normal sequence for P' can be refined to a one-sided sequence for P'.*

In particular

(c) *there exists a one-sided sequence for P', $\langle A_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$ such that $P' \subseteq \bigcup_{\alpha<\kappa} \underline{A_\alpha}$.*

Proof. (a) We build $\mathcal{A}$ by induction on $\alpha < \kappa$.

Induction base:

By Lemma 2.6, we may pick $A_0 \in [P']^{<\lambda}$ such that $\mathrm{cf}_P(A_0) > (\kappa+\lambda_0)^+$.

Inductive step:

Assume $\langle A_\beta \in [P']^{<\lambda} \mid \beta < \alpha\rangle$ has already been defined. Since $\alpha < \kappa = \mathrm{cf}(\lambda)$, we have $|V_\alpha| < \lambda$, thus, we may again apply Lemma 2.6 and pick $A_\alpha \in [P']^{<\lambda}$ with $\mathrm{cf}_P(A_\alpha) > (|V_\alpha| + \lambda_\alpha)^+$.

(b) Assume $\mathcal{A} = \langle A_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$ is a normal sequence for P'. Assume $\alpha < \kappa$, Set $B_\alpha := A_\alpha \setminus \underline{V_\alpha}$. Since $\mathrm{cf}_P(A_\alpha) > |V_\alpha| \geq \mathrm{cf}_P(\underline{V_\alpha})$, we have $\mathrm{cf}_P(B_\alpha) = \mathrm{cf}_P(A_\alpha)$. It follows that $\mathcal{B} = \langle B_\alpha \mid \alpha < \kappa\rangle$ is a one-sided sequence for P' refining $\mathcal{A}$.

(c) By the first item, we may pick a normal sequence for P', $\langle A_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$. Put $S := P' \setminus \bigcup_{\alpha<\kappa} A_\alpha$. Since $|S| \leq \lambda = \sup_{\alpha<\kappa} \mathrm{cf}_P(A_\alpha)$, we may partition $S = \bigcup_{\alpha<\kappa} S_\alpha$ such that $|S_\alpha| \leq \mathrm{cf}_P(A_\alpha)$ for all $\alpha < \kappa$, and then consider the normal sequence $\langle A_\alpha \cup S_\alpha \mid \alpha < \kappa\rangle$.

Repeating the proof of the second item, we end up with a sequence $\langle B_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$ satisfying the desired properties. □

Definition 3.6. *Assume $\langle P, \leq\rangle$ is a poset and $P' \in \mathrm{ESpec}_\lambda(P)$ for some singular cardinal λ. Assume $\mathcal{A} = \langle A_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$ is a one-sided sequence for P', where $\kappa = \mathrm{cf}(\lambda)$. $\mathcal{A}$ is said to be* upwards-extendible *iff for all $I \in [\kappa]^{<\kappa}$ and $X \subseteq^I \mathcal{A}$, there exists $\Sigma \in [\kappa]^\kappa$, such that $\langle A_\gamma \setminus \overline{X} \mid \gamma \in \Sigma\rangle$ is a one-sided sequence for P'.*

The next lemma justifies the definition of an upwards-extendible sequence.

Lemma 3.7. *Assume $\langle P, \leq\rangle$ is a poset and $P' \in \mathrm{ESpec}_\lambda(P)$ for some singular cardinal λ. If there exists an upwards-extendible one-sided sequence for P', then P' (and hence P) contains an antichain of size $\mathrm{cf}(\lambda)$.*

Proof. Put $\kappa := \mathrm{cf}(\lambda)$. Assume $\mathcal{A} = \langle A_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$ is an upwards-extendible sequence for P'. We build, by induction on $\alpha < \kappa$, two families $\{\tau_\alpha \mid \alpha < \kappa\}$ and $\{x_\alpha \in A_{\tau_\alpha} \mid \alpha < \kappa\}$.

Induction base:

Put $\tau_0 := 0$ and pick arbitrary $x_0 \in A_0$.

Inductive hypothesis:

Assume $I_\alpha := \{\tau_\beta \mid \beta < \alpha\}$ is increasing and $X_\alpha = \{x_\beta \in A_{\tau_\beta} \mid \beta < \alpha\}$ is an antichain.

Inductive step:

Since $\mathcal{A}$ is upwards-extendible and $I_\alpha \in [\kappa]^{<\kappa}$, there exists $\gamma > \sup(I_\alpha)$ such that $A_\gamma \setminus \overline{X_\alpha} \neq \varnothing$, so set $\tau_\alpha := \gamma$ and pick $x_\alpha \in A_\gamma \setminus \overline{X_\alpha}$.

To see that $X_\alpha \cup \{x_\alpha\}$ is an antichain, fix $\beta < \alpha$. By the choice of $x_\alpha \notin \overline{X_\alpha}$, clearly $x_\beta \nleq x_\alpha$. Since $\mathcal{A}$ is one-sided, we also have $x_\beta \ngeq x_\alpha$. This ends the construction. □

Here is a simple case where an upwards-extendible sequence must exist.

Theorem 3.8. *Assume $\langle P, \leq \rangle$ is a poset, and $\lambda > \mathrm{cf}(\lambda) = \kappa$ are cardinals. If there exists $P' \in \mathrm{ESpec}_\lambda(P)$, and $\mu < \lambda$, such that $|\underline{\{x\}} \cap P'| < \mu$ for all $x \in P'$, then there exists a one-sided sequence for P' which is upwards-extendible.*

Proof. We present an interesting proof, not necessarily the shortest one. Assume P' is like in the hypothesis. Let $\mathcal{A} := \langle A_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa \rangle$ be a one-sided sequence for P', with $|A_\alpha| > \mu$ for all $\alpha < \kappa$. Fix $\alpha < \kappa$.

Set $\lambda_\alpha := |V_\alpha|^+$. Put $C_\alpha := \{X \in [V_\alpha]^{<\kappa} \mid \mathrm{cf}_P(A_\alpha \setminus \overline{X}) \leq \lambda_\alpha\}$. If $C_\alpha = \varnothing$, let $D_\alpha := \varnothing$. Assume otherwise, let $D_\alpha \subseteq C_\alpha$ be a maximal subset of mutually disjoint sets. By $|D_\alpha| \leq |V_\alpha| < \lambda_\alpha$, regularity of λ_α, and the defining properties of C_α, we have $\mathrm{cf}_P(\bigcup\{A_\alpha \setminus \overline{X} \mid X \in D_\alpha\}) \leq \lambda_\alpha$. Since $\mathrm{cf}_P(A_\alpha) > \lambda_\alpha$ and $\mathrm{cf}_P(A_\alpha \setminus \bigcap\{\overline{X} \mid X \in D_\alpha\}) \leq \lambda_\alpha$, we conclude that $\bigcap\{\overline{X} \mid X \in D_\alpha\} \neq \varnothing$. Pick $y \in \bigcap\{\overline{X} \mid X \in D_\alpha\}$. Since members of D_α are mutually disjoint, $|D_\alpha| \leq |\underline{\{y\}} \cap P'| < \mu$. It follows that $|\bigcup D_\alpha| < \mu \cdot \kappa$.[2]

In both cases, we get that if $X \in [V_\alpha]^{<\kappa}$ and $(\bigcup D_\alpha) \cap X = \varnothing$, then $X \notin C_\alpha$. Set $D := \bigcup\{X \mid \exists \alpha < \kappa (X \in D_\alpha)\}$. Notice that $|D| \leq \mu \cdot \kappa$.

For $\alpha < \kappa$, set $B_\alpha := A_\alpha \setminus D$. It follows from $\mathrm{cf}_P(A_\alpha) > \mu \cdot \kappa \geq |D|$, that $\mathrm{cf}_P(B_\alpha) = \mathrm{cf}_P(A_\alpha)$, thus, $\mathcal{B} = \langle B_\alpha \mid \alpha < \kappa \rangle$ is a one-sided sequence for P'.

Finally, to see that $\mathcal{B}$ is upwards-extendible, pick $I \in [\kappa]^{<\kappa}$ and $X \subseteq^I \mathcal{B}$. Set $\Sigma := \kappa \setminus (\sup(I) + 1)$, and assume $\gamma \in \Sigma$. Since $X \cap D = \varnothing$, we have $X \notin C_\gamma$, and hence $\mathrm{cf}_P(A_\gamma \setminus \overline{X}) > \lambda_\gamma$. Since $\mathrm{cf}_P(A_\gamma \setminus B_\gamma) \leq |D| < \lambda_\gamma$, we must conclude that $\mathrm{cf}_P(B_\gamma \setminus \overline{X}) > \lambda_\gamma$. It now follows that $\langle B_\gamma \setminus \overline{X} \mid \gamma \in \Sigma \rangle$ is a one-sided sequence for P'. □

Theorem 3.9. *Assume $\langle P, \leq \rangle$ is a poset and $\lambda > \mathrm{cf}(\lambda) = \kappa$ are cardinals. If there is $P' \in \mathrm{ESpec}_\lambda(P)$ with $\mathrm{cf}_P(\{x \in P' \mid \mathrm{cf}_P(\overline{\{x\}} \cap P') < \lambda\}) = \lambda$, then there exists an antichain sequence for P' of length κ and cofinality λ.*

Proof. Let $\{x_i \mid i < \lambda\}$ be a bijective enumeration of P'. Let $\langle \lambda_\alpha \mid \alpha < \kappa \rangle$ be a strictly increasing sequence of cardinals cofinal in λ, with $\lambda_0 > \kappa$. Fix $\alpha < \kappa$ and set $A_\alpha := \{x_i \in P' \mid i < \lambda_\alpha, \mathrm{cf}_P(\overline{\{x_i\}} \cap P') < \lambda_\alpha\}$. Thus

$$\begin{aligned}\mathrm{cf}_P(\overline{A_\alpha} \cap P') &= \mathrm{cf}_P\left(\bigcup_{x \in A_\alpha} \overline{\{x\}} \cap P'\right) \\ &\leq \sum_{x \in A_\alpha} \mathrm{cf}_P(\overline{\{x\}} \cap P') \leq \lambda_\alpha \cdot \lambda_\alpha = \lambda_\alpha.\end{aligned}$$

Since $\{A_\alpha \mid \alpha < \kappa\}$ is an increasing chain of sets, each of bounded cardinality, and $\mathrm{cf}_P(\bigcup_{\alpha<\kappa} A_\alpha) = \mathrm{cf}_P(P') = \lambda$, the family $\Gamma := \{\mathrm{cf}_P(A_\alpha) \mid \alpha < \kappa\}$

[2] Notice this is the only usage of the hypothesis in the whole proof.

is unbounded in λ. Let $f : \kappa \to \kappa$ be the order-preserving injection defined recursively by letting $f(0) := \min\{\gamma < \kappa \mid \mathrm{cf}_P(A_\gamma) > \kappa\}$ and $f(\alpha) := \min\{\gamma < \kappa \mid \sum_{\beta<\alpha} \lambda_{f(\beta)} < \mathrm{cf}_P(A_\gamma)\}$ for α with $0 < \alpha < \kappa$.

To see that the definition is good, it suffices to recall that Γ is unbounded in λ, and to observe that, for all $\alpha < \kappa$, $\sum_{\beta<\alpha} \lambda_{f(\beta)}$ is bounded in λ, simply because $\alpha < \kappa = \mathrm{cf}(\lambda)$.

For all $\alpha < \kappa$, set $W_\alpha := \bigcup_{\beta<\alpha} A_{f(\beta)}$ and $B_\alpha := A_{f(\alpha)} \setminus (\underline{W_\alpha} \cup \overline{W_\alpha})$. To see that $\mathcal{B} := \langle B_\alpha \in [P']^{<\lambda} \mid \alpha < \kappa\rangle$ is an antichain sequence for P' of cofinality λ, we are left with showing that $\sup\{\mathrm{cf}_P(B_\alpha) \mid \alpha < \kappa\} = \lambda$. Fix $\alpha < \kappa$.

By $\mathrm{cf}_P(\underline{W_\alpha}) \le \mathrm{cf}_P(\overline{W_\alpha} \cap P') = \mathrm{cf}_P(\bigcup_{\beta<\alpha} \overline{A_{f(\beta)}} \cap P') \le \sum_{\beta<\alpha} \lambda_{f(\beta)}$ and the definition of f, we conclude that $\mathrm{cf}_P(A_{f(\alpha)}) > \mathrm{cf}_P((\underline{W_\alpha} \cup \overline{W_\alpha}) \cap P')$ and $\mathrm{cf}_P(B_\alpha) = \mathrm{cf}_P(A_{f(\alpha)})$. □

Even though it does not look so, it happens that the existence of an upwards-extendible sequence is equivalent to the existence of an antichain sequence. We prove this in Theorem 3.11 below.

Definition 3.10. *For a poset $\langle P, \le\rangle$, $P' \subseteq P$, $\mathrm{cf}_P(P') = \lambda$,$\mathrm{cf}(\lambda) = \kappa$, let*

$$\wp(P') := \{X \in [P']^{<\kappa} \mid \mathrm{cf}_P(P' \setminus \overline{X}) < \lambda\}.$$

Theorem 3.11. *Assume $\langle P, \le\rangle$ is a poset. For $\lambda > \mathrm{cf}(\lambda) = \kappa$, the following are equivalent:*

(a) *There exists $P_0 \in \mathrm{ESpec}_\lambda(P)$ with $\mathrm{cf}(\wp(P_0), \supseteq) = 0$.*
(b) *There exists $P_1 \in \mathrm{ESpec}_\lambda(P)$ with $\mathrm{cf}(\wp(P_1), \supseteq) \le \lambda$.*
(c) *There exists $P_2 \in \mathrm{ESpec}_\lambda(P)$ such that $\{P_2 \setminus \overline{X} \mid X \in \wp(P_2)\}$ is not $\subseteq$-cofinal in $\mathcal{I}_\lambda(P) \restriction P_2 = \{A \subseteq P_2 \mid \mathrm{cf}_P(A) < \lambda\}$.*
(d) *There exists $P_3 \in \mathrm{ESpec}_\lambda(P)$ and $\mathcal{A} = \langle A_\alpha \in [P_3]^{<\lambda} \mid \alpha < \kappa\rangle$, where $\mathcal{A}$ is a one-sided, upwards-extendible sequence for P_3.*
(e) *There exists $P_4 \in \mathrm{ESpec}_\lambda(P)$ and $\mathcal{B} = \langle B_\alpha \in [P_4]^{<\lambda} \mid \alpha < \kappa\rangle$, where $\mathcal{B}$ is an antichain sequence for $P_4 = \bigcup_{\alpha<\kappa} B_\alpha$.*

Proof. For the sake of this proof, fix a strictly increasing sequence of cardinals, $\langle \lambda_\alpha \mid \alpha < \kappa\rangle$, converging to λ.

(a $\Rightarrow$ b) This implication is trivial.

(e $\Rightarrow$ a) Let $\mathcal{B}$ be an antichain sequence for P_4 like in the hypothesis. Then by regularity of κ, for each $X \in [P_4]^{<\kappa}$ there exists some $\alpha > \kappa$ such that $X \subseteq \bigcup_{\beta<\alpha} B_\beta$, and it follows that $B_\delta \setminus \overline{X} = B_\delta$ for all $\delta > \alpha$, and hence $\mathrm{cf}_P(P_4 \setminus \overline{X}) \ge \sup\{\mathrm{cf}_P(B_\delta) \mid \alpha < \delta < \kappa\} = \lambda$, concluding that $\wp(P_4) = \varnothing$.

(b $\Rightarrow$ c) Let P_1 be like in the hypothesis. We claim that $P_2 := P_1$ works. Assume towards a contradiction that $\{P_2 \setminus \overline{X} \mid X \in \wp(P_2)\}$ is $\subseteq$-cofinal in $\mathcal{I}_\lambda(P) \restriction P_2$, then $\mathrm{cf}(\mathcal{I}_\lambda(P) \restriction P_2, \subseteq) \le \mathrm{cf}(\{\overline{X} \cap P_2 \mid X \in \wp(P_2)\}, \supseteq) \le \mathrm{cf}(\wp(P_1), \supseteq) \le \lambda$.

Clearly, $\mathcal{I}_\lambda(P) \restriction P_2 = \bigcup_{\alpha<\kappa}(\mathcal{I}_{\lambda_\alpha}(P) \restriction P_2)$. Also, by $P_2 \in \mathrm{ESpec}_\lambda(P)$, we have $\mathrm{cf}_P(P_2) = \lambda$ and $\mathrm{cov}(\mathcal{I}_{\lambda_\alpha}(P) \restriction P_2) = \lambda$ for all $\alpha < \kappa$. Thus, if indeed $\mathrm{cf}(\mathcal{I}_\lambda(P) \restriction P_2, \subseteq) \le \lambda$, then by Lemma 2.12, there exists some $X \in [P_2]^\kappa$ such that $X \notin \mathcal{I}_\lambda(P)$, which is an absurd.

(c $\Rightarrow$ d) Fix P_2 like in the hypothesis and let $Y \in \mathcal{I}_\lambda(P) \restriction P_2$ be a witness to the assumption. That is, $Y \not\subseteq P_2 \setminus \overline{X}$ for all $X \in \wp(P_2)$.

Let $P_3 := P_2 \setminus \underline{Y}$. By $\mathrm{cf}_P(Y) < \lambda$, we have $P_3 \in \mathrm{ESpec}_\lambda(P)$. By Lemma 3.5, let us fix a one-sided sequence for P_3, $\mathcal{A} := \langle A_\alpha \in [P_3]^{<\lambda} \mid \alpha < \kappa \rangle$ such that $P_3 \subseteq \underline{\bigcup_{\alpha<\kappa} A_\alpha}$.

To see that $\mathcal{A}$ is upwards-extendible, assume towards a contradiction that there exists some $I \in [\kappa]^{<\kappa}$ and $X \subseteq^I \mathcal{A}$ such that $\langle A_\alpha \setminus \overline{X} \mid \alpha \in \Sigma \rangle$ is not one-sided for all $\Sigma \in [\kappa]^\kappa$. It follows that $\mathrm{cf}_P(\bigcup_{\alpha<\kappa} A_\alpha \setminus \overline{X}) < \lambda$ and hence $\mathrm{cf}_P(P_2 \setminus \overline{X}) \leq \mathrm{cf}_P(\underline{Y} \cup P_3 \setminus \overline{X}) < \mathrm{cf}_P(Y)^+ + \lambda$, concluding that $X \in \wp(P_2)$.

However, by $Y \not\subseteq P_2 \setminus \overline{X}$ and $Y \subseteq P_2$, we get that $Y \cap \overline{X} \neq \varnothing$ and hence $\underline{Y} \cap X \neq \varnothing$, in contradiction with $X \subseteq^I \mathcal{A}$ and $\underline{Y} \cap A_\alpha = \varnothing$ for all $\alpha < \kappa$.

(d $\Rightarrow$ e) Assume $P_3 \in \mathrm{ESpec}_\lambda(P)$ and $\mathcal{A}$ given by the hypothesis. If there exists $Q \subseteq P_3$ such that $\mathrm{cf}_P(\{x \in Q \mid \mathrm{cf}_P(Q \cap \overline{\{x\}}) < \lambda\}) = \lambda$, then Theorem 3.9 completes the proof. Thus, from now on, we may assume

$$\mathrm{cf}_P(\{x \in Q \mid \mathrm{cf}_P(Q \cap \overline{\{x\}}) < \lambda\}) < \lambda \text{ for all } Q \subseteq P_3. \tag{3.1}$$

For all $\delta < \kappa$, denote $V^\delta := \bigcup\{A_\gamma \mid \delta \leq \gamma < \kappa\}$. Notice that by the hypothesis on $\mathcal{A}$, for all $\delta < \kappa$ and $X \subseteq^\delta \mathcal{A}$, we have $\mathrm{cf}_P(V^\delta \setminus \overline{X}) = \lambda$.

We now build the following objects by induction on $\alpha < \kappa$:

(i) Two increasing sequences of ordinals $I = \langle \tau_\alpha < \kappa \mid \alpha < \kappa \rangle$, and $J = \langle \delta_\alpha < \kappa \mid \alpha < \kappa \rangle$.
(ii) An antichain $X = \{x_\alpha \in A_{\tau_\alpha} \mid \alpha < \kappa\}$.
(iii) An antichain sequence of the form $\mathcal{B} = \langle B_\alpha \in [V^{\delta_\alpha}]^{<\lambda} \mid \alpha < \kappa \rangle$.

Induction base:

By $\mathrm{cf}_P(P_3) = \lambda$ and property (3.1), we may pick $\tau_0 < \kappa$ and $x_0 \in A_{\tau_0}$ such that $\mathrm{cf}_P(P_3 \cap \overline{\{x_0\}}) = \lambda$, hence, by Lemma 2.6 there exists $B_0 \in [P_3 \cap \overline{\{x_0\}}]^{<\lambda}$ with $\mathrm{cf}_P(B_0) > \lambda_0$. Put $\delta_0 := \tau_0$, and notice that since $\mathcal{A}$ is one-sided, indeed $B_0 \in [V^{\delta_0}]^{<\lambda}$

Induction hypothesis:

We assume the following objects have already been defined:

(i) Increasing $I_\alpha := \{\tau_\beta \mid \beta < \alpha\}$ and $J_\alpha := \{\delta_\beta \mid \beta < \alpha\}$.
(ii) An antichain $X_\alpha := \{x_\beta \in A_{\tau_\beta} \mid \beta < \alpha\}$.
(iii) A sequence $\langle B_\beta \in [V^{\delta_\beta} \cap \overline{\{x_\beta\}}]^{<\lambda} \mid \beta < \alpha \rangle$ such that
  - for all $\beta < \alpha$, $\lambda_\beta < \mathrm{cf}_P(B_\beta) < \lambda$ and
  - $\beta < \gamma < \alpha$ implies $\underline{B_\beta} \cap B_\gamma = \varnothing$ and $\overline{B_\beta} \cap B_\gamma = \varnothing$.

Inductive step:

By $I_\alpha \in [\kappa]^{<\kappa}$ and $\mathcal{A}$ being upwards-extendible, we may pick $\delta_\alpha > \sup(I_\alpha \cup J_\alpha)$ such that $\mathrm{cf}_P(V^{\delta_\alpha} \setminus \overline{X_\alpha}) = \lambda$. Set $Q := V^{\delta_\alpha} \setminus \overline{X_\alpha}$.

By (3.1), we choose $\tau_\alpha > \delta_\alpha$ and $x_\alpha \in (A_{\tau_\alpha} \setminus \overline{X_\alpha})$ such that $\mathrm{cf}_P(\overline{\{x_\alpha\}} \cap Q) = \lambda$. Set $\theta := \sum_{\beta<\alpha} |B_\beta|$. Since $|B_\beta| < \lambda$ for all $\beta < \alpha$, and $\alpha < \kappa = \mathrm{cf}(\lambda)$, we have $\theta < \lambda$. Thus, just like in the base case, we may find $B'_\alpha \in [Q \cap \overline{\{x_\alpha\}}]^{<\lambda}$ with $\mathrm{cf}_P(B'_\alpha) > \lambda_\alpha + \theta$. Finally, put $B_\alpha := B'_\alpha \setminus \underline{\bigcup_{\beta<\alpha} B_\beta}$. By $\mathrm{cf}_P(B'_\alpha) > \theta$, we conclude $\mathrm{cf}_P(B_\alpha) = \mathrm{cf}_P(B'_\alpha)$.

Pick $\beta < \alpha$. Clearly, $\underline{B_\beta} \cap B_\alpha = \varnothing$. To see that $\overline{B_\beta} \cap B_\alpha = \varnothing$, assume $y \in B_\beta, x \in B_\alpha$ with $x > y$. Since $y \in B_\beta \subseteq \overline{\{x_\beta\}}$, we must conclude that $x > x_\beta$. However, $B_\alpha \in [V^{\delta_\alpha} \setminus \overline{X_\alpha}]^{<\lambda}$ and $x_\beta \in X_\alpha$, in particular, $B_\alpha \subseteq P_3 \setminus \overline{\{x_\beta\}}$, therefore, $\overline{\{x_\beta\}} \cap B_\alpha = \varnothing$ and $x \not> x_\beta$. We conclude that $x \not> y$ and this ends the construction.

Evidently, the construction produces $\mathcal{B} := \langle B_\alpha \mid \alpha < \kappa \rangle$ which is an antichain sequence for $P_4 := \bigcup_{\alpha<\kappa} B_\alpha$. Finally, since P_3 is well-founded, $\sup\{\mathrm{cf}_P(B_\alpha) \mid \alpha < \kappa\} = \sup_{\alpha<\kappa} \lambda_\alpha = \lambda$, and $|B_\alpha| < \lambda$ for all $\alpha < \kappa$, we also have $P_4 \in \mathrm{ESpec}_\lambda(P)$. □

Thus, Corollary 2.20 may be improved to the following.

Corollary 3.12. *Assume* $\langle P, \leq \rangle$ *is a poset,* $\mathrm{cf}(P) = \lambda > \mathrm{cf}(\lambda) = \kappa$. *If there exists no antichain sequence of length* κ *and cofinality* λ *for* P*, then there exists* $\mathcal{S} \subseteq [\lambda]^{<\kappa}$ *such that* $\mathrm{cf}(\mathcal{S}, \supseteq) > \lambda$.

Corollary 3.13. *Assume* $\langle P, \leq \rangle$ *is a poset,* $\mathrm{cf}(P) = \lambda > \mathrm{cf}(\lambda) = \kappa$. *If any of the sets*

(a) $\{x \in P \mid \mathrm{cf}_P(\overline{\{x\}}) < \lambda\}$ *or*
(b) $\{x \in P \mid |\underline{\{x\}}| < \mu\}$ *for some* $\mu < \lambda$.

is of confinality λ*, then* P *contains* λ^κ *antichains of size* κ.

Proof. For (a), use Theorem 3.9 and Lemma 3.3. For (b), use Theorems 3.8, 3.11 and Lemma 3.3. □

It is worth mentioning that $\mathrm{cf}_P(\underline{\{x\}}) = 1$ for all $x \in P$. Also, we have already noticed in the proof of Lemma 2.17 that if $\mathrm{cf}(P) = \lambda$, then there exists $P' \in \mathrm{ESpec}_\lambda(P)$ such that $|\underline{\{x\}}| < \lambda$ for all $x \in P$.

Definition 3.14 ([8])**.** *Assume* $\langle P, \leq \rangle$ *is a poset.* $\langle P, \leq \rangle$ *is* cofinally homogeneous *iff for all* $x \in P$*,* $\mathrm{cf}_P(\overline{\{x\}}) = \mathrm{cf}(P)$.

Thus any updirected poset is cofinally homogeneous.

Definition 3.15. *Assume* $\langle P, \leq \rangle$ *is a poset.* $P' \subseteq P$ *is* externally homogeneous *iff for all* $x \in P'$*,*

$$\mathrm{cf}_P(P' \cap \overline{\{x\}}) = \mathrm{cf}_P(P' \setminus \overline{\{x\}}) = \mathrm{cf}_P(P').$$

The next corollary tells us that we may restrict our research to explore properties of externally homogeneous posets of singular cofinality.

Corollary 3.16. *Assume* $\langle P, \leq \rangle$ *is a poset,* $\kappa = \mathrm{cf}(\lambda) < \lambda \in \mathrm{ESpec}(P)$. *Then at least one of the following conditions holds:*

(a) *There exists an antichain sequence for* P *of length* κ *and cofinality* λ.
(b) *There exists an externally homogeneous poset* $P' \in \mathrm{ESpec}_\lambda(P)$.

Proof. Pick $P_1 \in \mathrm{ESpec}_\lambda(P)$. Set $P_2 := \{x \in P_1 \mid \mathrm{cf}_P(\overline{\{x\}} \cap P_1) < \lambda\}$ and $\mu := \mathrm{cf}_P(P_2)$. If $\mu = \lambda$ then (a) holds as a consequence of Theorem 3.9.

Assume $\mu < \lambda$. Set $P_3 := P_1 \setminus P_2$. Assume there exists $x \in P_3$ with $\mathrm{cf}_P(\overline{\{x\}} \cap P_3) = \theta < \lambda$. We show that x must be a member of P_2, contradicting $P_3 \cap P_2 = \varnothing$. Indeed, pick $A \in [P]^\theta$ such that $(\overline{\{x\}} \cap P_3) \subseteq \underline{A}$.

It follows that $(\overline{\{x\}} \cap P_1) \subseteq (\overline{\{x\}} \cap P_3) \cup (\overline{\{x\}} \cap P_2) \subseteq \underline{A \cup P_2}$, and $x \in P_2$, since $\mathrm{cf}_P(\overline{\{x\}} \cap P_1) \leq \theta + \mu < \lambda$. Thus, $\mathrm{cf}_P(\overline{\{x\}} \cap P_3) = \lambda$ for all $x \in P_3$.

Finally, put $P_4 := \{x \in P_3 \mid \mathrm{cf}_P(P_3 \setminus \overline{\{x\}}) < \lambda\}$. If $\mathrm{cf}_P(P_4) = \lambda$, then P_4 satisfies the hypothesis of Theorem 3.11.b and (a) holds. Otherwise, (b) holds for $P' := P_3 \setminus P_4$. □

We now revisit Lemma 3.5, claiming the existence of "principal" one-sided sequences.

Theorem 3.17. *Assume $\langle P, \leq \rangle$ is a poset, $\mathrm{cf}(P) = \lambda > \mathrm{cf}(\lambda) = \kappa$. Assume $\langle \lambda_\alpha \mid \alpha < \kappa \rangle$ is a sequence of cardinals cofinal in λ. Then at least one of the following conditions holds:*

- (a) *There exists an antichain sequence for P of length κ and cofinality λ.*
- (b) *There exists two families $\langle A_\alpha \in [P]^{<\lambda} \mid \alpha < \kappa \rangle$ and $\langle x_\alpha \in P \mid \alpha < \kappa \rangle$ such that*
  - (b.1) *for all $\alpha < \kappa$, $(\lambda_\alpha + \kappa + |V_\alpha|)^+ < \mathrm{cf}_P(A_\alpha) < \lambda$ and $A_\alpha \subseteq \overline{\{x_\alpha\}}$, where $V_\alpha := \bigcup_{\beta<\alpha} A_\beta$, and*
  - (b.2) *for all $\beta < \alpha < \kappa$, $A_\beta \cap \overline{\{x_\alpha\}} = \varnothing$.*

Proof. Assume $(\neg$a). By Corollary 3.16, we may pick an externally homogeneous $P' \in \mathrm{ESpec}_\lambda(P)$. We build the two families by induction on $\alpha < \kappa$.

<u>Induction base:</u>

Pick $x_0 \in P'$ arbitrarily. By $(\overline{\{x_0\}} \cap P') \in \mathrm{ESpec}_\lambda(P)$, we may use Lemma 2.6 to pick $A_0 \in [\overline{\{x_o\}} \cap P']^{<\lambda}$ such that $\mathrm{cf}_P(A_0) > (\lambda_0 + \kappa)^+$.

<u>Induction hypothesis:</u>

Assume $\langle x_\beta \in P' \mid \beta < \alpha \rangle$, $\langle A_\beta \in [\overline{\{x_\beta\}} \cap P']^{<\lambda} \mid \beta < \alpha \rangle$ have already been defined.

<u>Inductive step:</u>

Pick $x_\alpha \in P' \setminus \underline{V_\alpha}$ arbitrarily. Notice that by the choice of x_α, $(\overline{\{x_\alpha\}} \cap V_\alpha) = \varnothing$, and hence $A_\beta \cap \overline{\{x_\alpha\}} = \varnothing$ for all $\beta < \alpha$. Just like in the base case, since $(\overline{\{x_\alpha\}} \cap P') \in \mathrm{ESpec}_\lambda(P)$, we may pick $A_\alpha \in [\overline{\{x_\alpha\}} \cap P']^{<\lambda}$ such that $\mathrm{cf}_P(A_\alpha) > |(\lambda_\alpha + \kappa + |V_\alpha|)|^+$. This completes the construction.

□

4. Sufficient and equivalent conditions for the existence of antichain sequences

Theorem 4.1. *Assume $\langle P, \leq \rangle$ is a poset, and $\mathrm{cf}(P) = \lambda > \mathrm{cf}(\lambda) = \kappa$. If there exists $P' \in \mathrm{ESpec}_\lambda(P)$ such that either*

- (a) *$(\overline{\{x\}} \cap P')$ is linearly ordered for all $x \in P'$, or*
- (b) *$(\underline{\{x\}} \cap P')$ is linearly ordered for all $x \in P'$.*

then there exists an antichain sequence for P of length κ and cofinality λ.

Proof. (a) Recall that linear orders must have regular cofinality, and hence $\mathrm{cf}_P(\overline{\{x\}} \cap P') < \lambda$ for all $x \in P'$. The result now follows from Theorem 3.9.

(b) Let $\langle A_\alpha \mid \alpha < \kappa\rangle$ and $X = \{x_\alpha \mid \alpha < \kappa\}$ be as in Theorem 3.17.b. Put $B_\alpha := (A_\alpha \setminus \underline{X})$ for all $\alpha < \kappa$. Clearly, the refinement $\mathcal{B} = \langle B_\alpha \mid \alpha < \kappa\rangle$ is also a one-sided normal sequence for P. To see it is an antichain sequence, assume $\beta < \alpha < \kappa$ and $b \in B_\beta, a \in B_\beta$.

Since $\mathcal{B}$ is one-sided, we have $b \not\geq a$. Assume $b < a$. From $a \in A_\alpha \subseteq \overline{\{x_\alpha\}}$, we have $x_\alpha < a$. From $b \in A_\beta \subseteq \overline{\{x_\beta\}}$ and $b < a$, we have $x_\beta < b < a$. It follows that $\{x_\alpha, x_\beta, b\} \subseteq \underline{\{a\}}$, thus, by the hypothesis, $\{x_\alpha, x_\beta, b\}$ is linearly ordered by $<$. We now yield a contradiction, by showing that x_α and b are incomparable:

- since $b \in B_\beta \subseteq A_\beta$ and $(A_\beta \cap \overline{\{x_\alpha\}}) = \varnothing$, we have $b \not\geq x_\alpha$.
- since $b \in B_\beta = A_\beta \setminus \underline{X} \subseteq (A_\beta \setminus \underline{\{x_\alpha\}})$, we have $b \not\leq x_\alpha$.

□

Recall that a poset $\langle T, \leq\rangle$ is a *pseudotree* iff $\langle \underline{\{x\}}, \leq\rangle$ is linearly-ordered for all $x \in T$.

Corollary 4.2. *A counter-example to the Milner-Sauer conjecture cannot embed a pseudotree of the same cofinality.*

Thus, if for a cardinal λ, we denote by $\mathcal{T}_\lambda$ the class of all λ-Aronszajn trees, then $\forall\lambda(\lambda > \mathrm{cf}(\lambda) \to \mathrm{MS}(\mathcal{T}_\lambda))$ holds.

Recall that for a sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$, and $\alpha < \kappa$, we let $V_\alpha := \bigcup_{\beta<\alpha} A_\alpha$ and $V^\alpha := \bigcup_{\alpha\leq\gamma<\kappa} A_\gamma$. In the following two theorems, we show that normal sequences with untight mutual relations between their components can be refined into sequences with no mutual relations at all.

Theorem 4.3 (Upward boundness properties)**.** *Assume $\langle P, \leq\rangle$ is a poset, and $\lambda > \mathrm{cf}(\lambda) = \kappa$ are cardinals. The following are equivalent :*

1. *There exists $P_1 \in \mathrm{ESpec}_\lambda(P)$ and an antichain sequence for P_1 of length κ and cofinality λ.*
2. *There exists $P_2 \in \mathrm{ESpec}_\lambda(P)$ and $\mathcal{A} = \langle A_\alpha \in [P_2]^{<\lambda} \mid \alpha < \kappa\rangle$ which is a normal sequence, $\mathrm{cf}_P(\bigcup_{\alpha<\kappa} A_\alpha) = \lambda$, and any of the following conditions hold:*
   (2.1) $\sup\{|V^\alpha \cap \overline{V_\alpha}| \mid \alpha < \kappa\} < \lambda$,
   (2.2) $\sup\{|V^\alpha \cap \overline{\{x\}}| \mid x \in V_\alpha, \alpha < \kappa\} < \lambda$, *or*
   (2.3) $\sup\{\mathrm{cf}_P(V^\alpha \cap \overline{\{x\}}) \mid x \in V_\alpha, \alpha < \kappa\} < \lambda$.

Proof. Obviously, $(1 \Rightarrow 2.1 \Rightarrow 2.2 \Rightarrow 2.3)$ as $V^\alpha \cap \overline{V_\alpha} = \varnothing$ for all $\alpha < \kappa$, whenever $\mathcal{A}$ is an antichain sequence. We show $(2.3 \Rightarrow 1)$.

Set $P' := \bigcup\{A_\alpha \mid \alpha < \kappa\}$ and $\mu := \sup\{\mathrm{cf}_P(V^\alpha \cap \overline{\{x\}}) \mid x \in V_\alpha, \alpha < \kappa\}$. Fix $x \in P'$. Find $\alpha < \kappa$ such that $x \in A_\alpha$. We have

$$\begin{aligned}\mathrm{cf}_P(\overline{\{x\}} \cap P') &\leq \mathrm{cf}_P(\overline{\{x\}} \cap V_{\alpha+1}) + \mathrm{cf}_P(\overline{\{x\}} \cap V^{\alpha+1}) \\ &\leq \mathrm{cf}_P(V_{\alpha+1}) + \mu < \lambda.\end{aligned}$$

The implication now follows from Theorem 3.9. □

Theorem 4.4 (Downward boundness properties)**.** *Assume $\langle P, \leq\rangle$ is a poset, and $\lambda > \mathrm{cf}(\lambda) = \kappa$ are cardinals. The following are equivalent :*

(1) *There exists $P_1 \in \mathrm{ESpec}_\lambda(P)$ and an antichain sequence for P_1 of length κ and cofinality λ.*

(2) *There exists $P_2 \in \mathrm{ESpec}_\lambda(P)$ and $\mathcal{A} = \langle A_\alpha \in [P_2]^{<\lambda} \mid \alpha < \kappa\rangle$ which is a normal sequence, $\mathrm{cf}_P(\bigcup_{\alpha<\kappa} A_\alpha) = \lambda$, and any of the following conditions hold:*

(2.1) $\sup\{|V_\alpha \cap \underline{A_\alpha}| \mid \alpha < \kappa\} < \lambda$,

(2.2) $\sup\{|V_\alpha \cap \underline{\{x\}}| \mid x \in A_\alpha, \alpha < \kappa\} < \lambda$, *or*

(2.3) $\sup\{\mathrm{cf}_P(V_\alpha \cap \underline{A_\alpha}) \mid \alpha < \kappa\} < \lambda$.

Proof. Evidently, ($1 \Rightarrow 2.1 \Rightarrow 2.2$) and ($1 \Rightarrow 2.3$). By Theorems 3.8 and 3.11, we have ($2.2 \Rightarrow 1$). We are left with showing ($2.3 \Rightarrow 1$).

Assume $\mathcal{A} = \langle A_\alpha \in [P_2]^{<\lambda} \mid \alpha < \kappa\rangle$ is the normal sequence given by the hypothesis. Put $\mu := \sup\{\mathrm{cf}_P(V_\alpha \cap \underline{A_\alpha}) \mid \alpha < \kappa\}$ and $\lambda_\alpha := \mathrm{cf}_P(A_\alpha)$ for all $\alpha < \kappa$. By Lemma 3.5 and $\mu, \kappa < \lambda$, we may also assume that $\mathcal{A}$ is one-sided and $\lambda_0 > \mu + \kappa$.

For each $\alpha < \kappa$, let $X_\alpha \in [P]^{\leq\mu}$ be such that $V_\alpha \cap \underline{A_\alpha} \subseteq \underline{X_\alpha}$. Define $C_\alpha := \bigcup\{A_\alpha \cap \underline{X_\delta} \mid \alpha < \delta < \kappa\}$, and $B_\alpha := A_\alpha \setminus C_\alpha$.

To see that $\mathcal{B} = \langle B_\alpha \mid \alpha < \kappa\rangle$ is an antichain sequence, let $y \in B_\beta, x \in B_\alpha$ for some $\beta < \alpha < \kappa$. Since $V_\alpha \cap \underline{\{x\}} \subseteq V_\alpha \cap \underline{A_\alpha} \subseteq \underline{X_\alpha}$ and $B_\beta \cap \underline{X_\alpha} = \varnothing$, it follows that $y \not\leq x$. Since $\mathcal{A}$ is a one-sided sequence, we also have $x \not\leq y$, therefore x and y are incomparable. Finally, to evaluate the cofinality of $\mathcal{B}$, fix $\alpha < \kappa$ and observe that $A_\alpha = B_\alpha \cup C_\alpha$, $\mathrm{cf}_P(A_\alpha) = \lambda_\alpha$, while

$$\mathrm{cf}_P(C_\alpha) \leq \mathrm{cf}_P\left(\bigcup\{X_\delta \mid \alpha < \delta < \kappa\}\right) \leq \mu \cdot \kappa < \lambda_\alpha.$$

Hence $\mathrm{cf}(B_\alpha) = \lambda_\alpha$ and $\lambda = \sup\{\lambda_\alpha \mid \alpha < \kappa\} \leq \mathrm{cf}_P(\bigcup_{\alpha<\kappa} B_\alpha) \leq \lambda$. □

Definition 4.5. *Assume a poset $\langle P, \leq\rangle$, $P' \subseteq P$, and $\mathcal{I}$ is an ideal over P'. We say that $\mathcal{I}$* is unbounded in *P' iff there exists an increasing $\subseteq$-chain $\{U_\alpha \mid \alpha < \mu\} \subseteq \mathcal{I}$ such that $\sup\{\mathrm{cf}_P(U_\alpha) \mid \alpha < \mu\} = \mathrm{cf}_P(P')$.*

Corollary 4.6. *Assume $\langle P, \leq\rangle$ is a poset, $P' \in \mathrm{ESpec}_\lambda(P)$ for some singular cardinal λ. Then, the ideal $[P']^{<\lambda}$ is unbounded in P'.*

Proof. We repeat the proof of Lemma 2.6.a. Put $\kappa := \mathrm{cf}(\lambda)$. Let $P' = \bigcup_{\alpha<\kappa} U_\alpha$, where $\{U_\alpha \mid \alpha < \kappa\} \subseteq [P']^{<\lambda}$ is an increasing $\subseteq$-chain. It follows that $\lambda = \mathrm{cf}_P(P') = \mathrm{cf}_P\left(\bigcup_{\alpha<\kappa} U_\alpha\right) \leq \sum_{\alpha<\kappa} \mathrm{cf}_P(U_\alpha)$ and hence $\sup\{\mathrm{cf}_P(U_\alpha) \mid \alpha < \kappa\} = \lambda$. □

Definition 4.7. *Assume $\langle P, \leq\rangle$ is a poset, $P' \subseteq P$ and $\mathrm{cf}_P(P') = \lambda$. Define the ideal $\mathcal{J}_\mu(P') := \{Y \in [P']^{<\lambda} \mid \mathrm{cf}(\mathcal{I}_\mu(P') \restriction Y, \subseteq) < \lambda\}$, where μ denotes a cardinal and $\mathcal{I}_\mu(P') \restriction Y := \{X \subseteq Y \mid \mathrm{cf}_{P'}(X) < \mu\}$.*

For $\kappa = \mathrm{cf}(\lambda)$, it is not hard to see that assuming the SSH, the ideal $\mathcal{J}_\kappa(P')$ is κ-complete. Consequently, in this case, $\mathcal{J}_\kappa(P')$ is unbounded in P' iff $\sup\{\mathrm{cf}_P(U) \mid U \in \mathcal{J}_\kappa(P')\} = \lambda$.

For notational simplicity, we shall further say that $\mathcal{J}(P')$ is unbounded in P' to express: $\exists\mu < \mathrm{cf}_P(P')$ ($\mathcal{J}_\mu(P')$ is unbounded in P').

Lemma 4.8. *Assume $\langle P, \leq\rangle$ is a poset, $P' \in \mathrm{ESpec}_\lambda(P)$ for a cardinal λ. If λ is a singular strong limit cardinal, then $\mathcal{J}(P')$ is unbounded in P'.*

Proof. Since λ is a strong limit, $\mathrm{cf}(\mathcal{I}_\omega(P') \restriction Y, \subseteq) \leq 2^{|Y|} < \lambda$ for all $Y \in [P']^{<\lambda}$. Thus, $\mathcal{J}_\omega(P') = [P']^{<\lambda}$ and we may appeal to Corollary 4.6. $\square$

The proof of the next theorem was inspired by a beautiful idea from [5].

Theorem 4.9 (Boundness by ideals)**.** *Assume $\langle P, \leq\rangle$ is a poset and $\lambda > \mathrm{cf}(\lambda) = \kappa$ are cardinals. The following are equivalent:*

(a) *There exists $P' \in \mathrm{ESpec}_\lambda(P)$ such that $\mathcal{J}(P')$ is unbounded in P'.*
(b) *There exists $P'' \in \mathrm{ESpec}_\lambda(P)$ and $\mathcal{A} = \langle A_\alpha \in [P'']^{<\lambda} \mid \alpha < \kappa\rangle$ which is an antichain sequence of length κ and cofinality λ.*

Proof. To help the reader get used to the definition, we start with (b $\Rightarrow$ a). Assume $\mathcal{A} = \langle A_\alpha \in [P'']^{<\lambda} \mid \alpha < \kappa\rangle$ is like in (b). Put $P' := \bigcup_{\alpha<\kappa} A_\alpha$. To see that $\mathcal{J}(P')$ is unbounded in P', it suffices to show that $\{V_\alpha \mid \alpha < \kappa\} \subseteq \mathcal{J}_\omega(P')$. Let $\alpha < \kappa$, we need to verify that $\mathrm{cf}(\mathcal{I}_\omega(P') \restriction V_\alpha, \subseteq) < \lambda$.

Evidently, $\mathcal{I}_\omega(P') \restriction V_\alpha$ is generated by $\{\underline{X} \cap V_\alpha \mid X \in [P']^{<\omega}\}$. By $\mathcal{A}$ being an antichain sequence, it is actually generated by $\{\underline{X} \cap V_\alpha \mid X \in [V_\alpha]^{<\omega}\}$. It follows that $\mathrm{cf}(\mathcal{I}_\omega(P') \restriction V_\alpha, \subseteq) \leq |[V_\alpha]^{<\omega}| = |V_\alpha| < \lambda$.

(a $\Rightarrow$ b) Assume P' is like in (a), $\theta < \lambda$ and $\{U_\alpha \mid \alpha < \kappa\} \subseteq \mathcal{J}_\theta(P')$ is a witness to the unboundedness property. We define by induction an increasing function $f : \kappa \to \kappa$ and a normal sequence $\mathcal{A} = \langle A_\alpha \subseteq U_{f(\alpha)} \mid \alpha < \kappa\rangle$ with property (2.3) of Theorem 4.4.

<u>Induction base:</u>

Let $f(0) := \min\{\gamma < \kappa \mid \mathrm{cf}_P(U_\gamma) > \kappa\}$ and $A_0 := U_{f(0)}$.

<u>Induction hypothesis:</u>

Assume $f \restriction \alpha$ and $\langle A_\beta \mid \beta < \alpha\rangle$ have already been defined, such that $\mathrm{cf}_P(V_\beta \cap \underline{A_\beta}) < \theta$ for all $\beta < \alpha$.

<u>Inductive step:</u>

Put $\tau_\alpha := \sup_{\beta<\alpha} f(\beta)$ and $\mu_\alpha := \mathrm{cf}(\mathcal{I}_\theta(P') \restriction V_\alpha, \subseteq)$. By $V_\alpha \subseteq U_{\tau_\alpha} \in \mathcal{J}_\theta(P')$, we have $V_\alpha \in \mathcal{J}_\theta(P')$ and $\mu_\alpha < \lambda$.

Fix $\{Y_i \in [P']^{<\theta} \mid i < \mu_\alpha\}$ such that $\{V_\alpha \cap \underline{Y_i} \mid i < \mu_\alpha\}$ is indeed cofinal in $\mathcal{I}_\theta(P') \restriction V_\alpha$. Put $f(\alpha) := \min\{\gamma < \kappa \mid (|U_{\tau_\alpha}| + \mu_\alpha)^+ < \mathrm{cf}_P(U_\gamma)\}$.

Now, for all $i < \mu_\alpha$, set $B_i := \{x \in U_{f(\alpha)} \mid V_\alpha \cap \underline{\{x\}} \subseteq V_\alpha \cap \underline{Y_i}\}$. Since $U_{f(\alpha)} \subseteq P'$, we have $U_{f(\alpha)} = \bigcup\{B_i \mid i < \mu_\alpha\}$. By $\mathrm{cf}_P(U_{f(\alpha)}) > (|U_{\tau_\alpha}| + \mu_\alpha)^+$, there must exist some $j < \mu$ such that $\mathrm{cf}_P(B_j) > |U_{\tau_\alpha}|^+$. Set $A_\alpha := B_j$.

Finally, since $(V_\alpha \cap \underline{A_\alpha}) \subseteq \underline{Y_j}$, we have $\mathrm{cf}_P(V_\alpha \cap \underline{A_\alpha}) \leq |Y_j| < \theta$. $\square$

We end with a corollary to [5] or to the preceding theorem.

Corollary 4.10. *Suppose λ is a singular cardinal and MS_λ fails. Then there exists a poset $\langle P, \leq\rangle$ of cofinality λ with no antichains of size $\mathrm{cf}(\lambda)$*

and no chains of size λ. In fact, this $\langle P, \leq\rangle$ satisfies

$$\sup\{|L| \mid L \subseteq P \text{ is linearly-ordered }\} < \lambda.$$

Proof. Let $\langle Q, \leq\rangle$ witness $\neg\,\mathrm{MS}_\lambda$. Pick $Q' \in \mathrm{ESpec}_\lambda(Q)$. Put $\kappa := \mathrm{cf}(\lambda)$ and take an increasing $\subseteq$-chain of sets $\{A_\alpha \mid \alpha < \kappa\} \subseteq [Q']^{<\lambda}$ such that $\left(\bigcup_{\alpha<\kappa} A_\alpha\right) = Q'$. By the preceding theorem, $\mathcal{J}_\omega(Q')$ is not unbounded in Q', and hence there is some $\alpha < \kappa$ such that $A_\alpha \notin \mathcal{J}_\omega(Q')$. Fix such α.

Put $P' := \{A_\alpha \cap \underline{\{x\}} \mid x \in Q'\}$. Since

$$\mathrm{cf}\left(\{X \subseteq A_\alpha \mid \mathrm{cf}_{Q'}(X) < \omega\}, \subseteq\right) \;=\; \lambda,$$

we have $\mathrm{cf}(P', \subseteq) = \lambda$, so let us pick $P \in \mathrm{ESpec}_\lambda(P')$. We claim that $\langle P, \subseteq\rangle$ works. Indeed, P cannot contain an antichain of size κ, since Q' does not contain one. Finally, $\langle P, \subseteq\rangle$ is well-founded, hence, for any linearly-ordered $L \subseteq P$, we have $|L| \leq |\bigcup L|$. It follows that

$$\sup\{|L| \mid L \subseteq P \text{ is linearly-ordered }\} \leq |A_\alpha| < \lambda.$$

□

5. Acknowledgements

This paper, together with [12], covers the main results from the author's M.Sc. thesis [11] written at the Tel-Aviv University under the supervision of M. Gitik to whom he wishes to express his deep gratitude and appreciation.

The author would also like to thank M. Pouzet and I. Gorelic for introducing him to the subject of partial orders with singular cofinality.

We are grateful to A. Shomrat for the mathematical cooperation throughout the last couple of years, and his comments on this paper.

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School of Mathematical Sciences, Tel Aviv University
Tel Aviv 69978, Israel
E-mail address: assaf@rinot.com