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TWO-CHARACTER SETS IN FINITE LINEAR SPACES

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Dedicated to the centenary of the birth of Ferenc Kárteszi (1907–1989).

Abstract. A set of type (m, n) $\mathcal{K}$ is a set of points of a finite linear space with the property that each line of the linear space meets $\mathcal{K}$ in either m or n points. In this paper, sets of type (m, n) in finite linear spaces with constant point degree are studied, and some characterization results are given.

1. Introduction

A (*finite*) *linear space* is a pair $\mathbf{S} = (\mathcal{P}, \mathcal{L})$ consisting of a (finite) set $\mathcal{P}$ of elements called *points* and a set $\mathcal{L}$ of distinguished subsets of points, called *lines*, such that *any two distinct points are contained in exactly one line, any line has at least two points, and there are at least two lines.*

Let $\mathbf{S}$ be a finite linear space. We use v and b to denote the number of points and of lines of $\mathbf{S}$, respectively. For any point p, the *degree* of p is the number r_p of lines on p, and for any line L, the *length* of L is the number k_l of its points.

Every 2–$(v, k, 1)$–design is a finite linear space with constant point degree $r = (v-1)/(k-1)$. Tallini [11] defines k–sets of type $(m_1, \ldots, m_s)$ in a 2–$(v, k, 1)$–design and proves some necessary conditions for their existence. He also posed the problem of investigating such sets and, ever since in the literature one can find some papers on sets of type $(m_1, \ldots, m_s)$ in 2–(v, k, λ)–designs, $\lambda \geq 1$, see e.g. [3, 4, 8, 6].

In this paper, we define sets of type $(m_1, \ldots, m_s)$ in a finite linear space with constant point degree r, we study finite linear spaces with $r = q + 1$ containing a set of type (m, n) (Section 2), and finite planar spaces whose planes pairwise intersect each other either in the empty–set or in a line, and with a set of type $(0, n)$ (Section 3). As a consequence of these results we obtain a new proof of a result of Scafati [9] on sets of type $(0, n)$ in $\mathrm{P}G(r, q)$.

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A *k–set* of a finite linear space $\mathbf{S}$ is a subset of size k of the point–set of $\mathbf{S}$.

Let $\mathbf{S}$ be a finite linear space with constant point degree r, every line of $\mathbf{S}$ has at most r points. Let $r-h$ and $r-H$, $h, H \geq 0$, denote the maximal and minimal length for the lines of $\mathbf{S}$ respectively, and let $m_1 < m_2 < \ldots < m_s$, be s (≥ 1) non–negative integers with $m_s \leq r-h$. A *k–set* $\mathcal{K}$ of $\mathbf{S}$ is of *class* $[m_1, \ldots, m_s]$ if each line of $\mathbf{S}$ meets $\mathcal{K}$ in $m_1, m_2, \ldots,$ or m_s points. Let t_{m_j} be the number of lines meeting $\mathcal{K}$ in exactly m_j points. A k–set $\mathcal{K}$ is of type $(m_1, \ldots, m_s)$ if it is of class $[m_1, \ldots, m_s]$ and $t_{m_j} \neq 0$ for every $j = 1, \ldots, s$. The m_j's are the *characters* of $\mathbf{S}$.

A line which meets $\mathcal{K}$ in i points is called an *i–secant*. A 1–secant is called a *tangent*.

The one–character case is easily solved.

Proposition 1.1. *A subset $\mathcal{K}$ of type (m) of the point–set of a finite linear space $\mathbf{S}$ with constant point degree, is either the empty set or $\mathbf{S}$.*

Proof. If $\mathcal{K}$ is neither the empty set nor $\mathbf{S}$, then there are a point $p \in \mathcal{K}$ and a point $q \notin \mathcal{K}$.

Let k denote the size of $\mathcal{K}$. Counting k via the lines through p gives $k = r(m-1)+1$. On the other hand counting k via the lines through q gives $k = rm$, and comparing the two values obtained for k one gets a contradiction. $\square$

In this paper k–sets of type[1] (m, n), $0 \leq m < n \leq r-h$, will be considered. In particular, the following result will be proved.

Theorem 1.2. *Let $\mathcal{K}$ be a k–set of type $(0, q)$ of $\mathbf{S}$. Then, there is a unique line ℓ disjoint with $\mathcal{K}$ and either $\mathbf{S}$ is a projective plane of order q and $\mathcal{K}$ is the affine restriction of $\mathbf{S}$ with respect to ℓ, or $\mathbf{S}$ is obtained from a projective plane π by deleting t collinear points on a line disjoint with a set of type $(0, q)$ of π.*

Definition 1.3. *Let $\mathbf{S}$ be a finite linear space with constant point degree $q+1$ and q square. A* Baer subplane *of $\mathbf{S}$ is a $(q+\sqrt{q}+1)$–set of type $(1, 1+\sqrt{q})$.*

Theorem 1.4. *Let $\mathbf{S}$ be a finite linear space with constant point degree $q+1$, and let $\mathcal{K}$ be a $(q+n)$–set of type $(1, n)$ of $\mathbf{S}$, with $q \leq (n-1)^2$. Then, q is a square and $n = \sqrt{q}+1$. Furthermore, if the lines of $\mathbf{S}$ have constant length, then $\mathbf{S}$ is a projective plane of order q, and $\mathcal{K}$ is a Baer subplane of $\mathbf{S}$.*

Definition 1.5. *A* unital *of a finite linear space with constant point degree $q+1$ and q square, is a $(q\sqrt{q}+1)$–set of type $(1, 1+\sqrt{q})$.*

Theorem 1.6. *Let $\mathbf{S}$ be a finite linear space with constant point degree $q+1$, $b = q^2+q+1$ lines, and let $\mathcal{K}$ be a set of type $(1, n)$ of maximal size $(n-1)q+1$ in $\mathbf{S}$. Then $q = (n-1)^2$. Moreover, if all the lines of $\mathbf{S}$ have*

[1] Clearly, $m \leq r-H$.

constant length, then **S** *is a projective plane of order* q *and* $\mathcal{K}$ *is a unital of* **S**.

A *subspace* of a linear space is a subset of points X such that for every pair of distinct points of X the line joining them is entirely contained in X.

A *finite planar space* is a linear space endowed with a family of subspaces, called *planes*, such that any three non–collinear points are contained in a unique plane, every plane contains at least three non–collinear points, and there are at least two planes.

Clearly, projective and affine spaces of dimension at least 3 are planar spaces.

Theorem 1.7. *Let* **S** *be a planar space whose planes pairwise intersect either in the empty set or in a line, and let* $\mathcal{K}$ *be a set of type*[2] $(0, n)_1$ *in* **S**, *then* $n = 1$ *or* $n = q$. *If* $n = 1$, $\mathcal{K}$ *is a point. If* $n = q$, *there are lines of length* $q + 1$, **S** *is obtained from* $\mathrm{PG}(3, q)$ *by deleting a subset* X *of points, and* $\mathcal{K}$ *is the complement of a* (*possibly punctured*) *plane* π *of* **S**.

1.1. **Some preliminary results.** This section contains some results on finite linear spaces, and two–character sets of a finite projective plane, some of which will be useful in the next sections.

1.1.1. *Linear spaces.*

Definition 1.8. *Let* **S** *be a finite linear space, and* H *be a finite set of non–negative integers.* **S** *is* H–semiaffine *if for every point–line pair* (p, ℓ), *with* $p \notin \ell$, *the number* $\pi(p, \ell) := r_p - k_\ell \in H$.

Theorem 1.9 (Beutelspacher–Kersten [1])**.** *Let* **S** *be a finite* $\{0, 1, t\}$*–semiaffine linear space with maximum point degree* $q+1$. *Then* **S** *is the complement of a set of points* X *in a projective plane of order* q, *such that each line of the plane meets* X *in* 0, 1, *or* t *points.*

Theorem 1.10 (Metsch [7])**.** *Suppose that* **S** *is a locally projective planar space of order* $q \geq 2$ *with at least* $q^3 - q^2 + q + 2$ *points. Then* q *is a prime power and* **S** *can be embedded in* $\mathrm{PG}(3, q)$.

1.1.2. *Two–character sets in a projective plane.* For a list of most of the papers devoted to k–sets we refer the reader to [10]. Here, we recall only two results of Tallini on (m, n)–sets in a projective plane.

Theorem 1.11 (Tallini [13])**.** *Let* π_q *be a projective plane of order* q *with* $q/(n-1) = p^h$, p *a prime,* h *a positive integer, and let* $\mathcal{K}$ *be a set of type* $(1, n)$ *in* π_q. *Then,* $q = p^{2h}$, $n = 1 + \sqrt{q}$ *and* $\mathcal{K}$ *is either a Baer subplane or a unital.*

[2]Note that, as usual in projective geometry, to denote a set of type (m, n) with respect to the lines of a linear space of dimension $d \geq 3$, we use the notation $(m, n)_1$.

Theorem 1.12 (Tallini [13]). *Let π_q be a projective plane of order q and $\mathcal{K}$ be a k–set of type (m, n) in π_q. If $(n-m)^2 \geq q$, $\gcd(m, n) = \gcd(m-1, n-1) = 1$, then q is a square, $n = m + \sqrt{q}$ and either $k = m(q + \sqrt{q} + 1)$ or $k = (m + \sqrt{q})(q - \sqrt{q} - 1)$.*

2. Two–character sets in a finite linear space with constant point degree $q + 1$

In this section we are going to study sets of type (m, n) in a finite linear space with constant point degree $q+1$ and $b = q^2+q+1+z$ lines. First we will study $(0, n)$–sets, then we will give the basic equations for the characters of a set of type (m, n); in particular, we study these sets for $m = 1$, characterizing them in the case of extremal sizes.

Throughout this section, let $\mathbf{S} = (\mathcal{P}, \mathcal{L})$ be a finite linear space with constant point degree $q + 1$, and let $q + 1 - h$ and $q + 1 - H$ denote the maximal and minimal length for the lines of $\mathbf{S}$, respectively.

Clearly, $2 \leq q + 1 - H \leq q + 1 - h \leq q + 1$. If $m = q + 1$, then $n > q + 1 \geq q + 1 - h$, a contradiction. Thus, $0 \leq m \leq q$. If $n = q + 1$, then $q + 1 - h = q + 1$, and so $b = q^2 + q + 1$.

Proposition 2.1. *A set $\mathcal{K}$ of type $(0, 1)$ is a point of $\mathbf{S}$.*

Proposition 2.2. *Let $\mathbf{S}$ be a finite linear space with constant point degree $q + 1$. Then, there is no set of type $(0, q + 1)$ in $\mathbf{S}$. Moreover, if $\mathbf{S}$ contains a set $\mathcal{K}$ of type $(q, q + 1)$, then $\mathbf{S}$ is a projective plane of order q and $\mathcal{K}$ is the complement of a point in $\mathbf{S}$.*

Proof. A line of length $q + 1$ meets every other line of the linear space, and since a set of type $(0, q + 1)$ contains a line of length $q + 1$, the first part of the statement easily follows.

If $\mathcal{K}$ is a set of type $(q, q + 1)$, then it does not coincide with $\mathbf{S}$, hence there is a point $x \in \mathbf{S} \setminus \mathcal{K}$. Every line through x has length $q + 1$, and so $v = q^2 + q + 1 = b$, that is $\mathbf{S}$ is a projective plane and $\mathcal{K}$ is the complement of x. □

2.1. k–sets of type $(0, n)$.

In view of the previous results, we may assume that $2 \leq n \leq q$.

Proposition 2.3. *Let $\mathcal{K}$ be a k–set of type $(0, n)$ in $\mathbf{S}$. Then $n \mid q$.*

Proof. Counting k via the lines through a point P of $\mathcal{K}$ gives

$$k = (q + 1)(n - 1) + 1 = q(n - 1) + n. \tag{2.1}$$

If Q is a point outside K, let r denote the number of lines through Q which have n points in common with K. Then

$$rn = k = q(n - 1) + n,$$

and so

$$r = q + 1 - q/n, \tag{2.2}$$

from which the assertion follows. □

Theorem 2.4. *Let $\mathcal{K}$ be a k–set of type $(0, q)$ of $\mathbf{S}$. Then there is a unique line disjoint from $\mathcal{K}$, and either $\mathbf{S}$ is a projective plane of order q and $\mathcal{K}$ is the affine plane $\mathbf{S} \setminus \ell$, or $\mathbf{S}$ is obtained from a projective plane π by deleting t collinear points on a line disjoint from a set of type $(0, q)$ of π.*

Proof. In such a case $q+1-h \geq q$. Since through every point not in $\mathcal{K}$ there is at least one line meeting $\mathcal{K}$ in q points, and so it has length $q+1$. Thus, there are lines of length $q+1$. More precisely, from (2.2) it follows that on a point outside $\mathcal{K}$ there are q lines of length $q+1$, and one line disjoint from $\mathcal{K}$. Therefore, all the lines disjoint from $\mathcal{K}$ have the same length, say $q+1-t$. Thus, $|\mathbf{S}| = q^2+q+1-t$.

If $t = 0$, then $\mathbf{S}$ is a projective plane of order q, and $\mathcal{K}$ is the complement of a line, that is, an affine plane.

Now, let $t \geq 1$. If all the lines through a point of $\mathcal{K}$ have length q, then $|\mathbf{S}| = q^2$, a contradiction. So, through each point of $\mathcal{K}$ there are both lines of length q and $q+1$.

From $|\mathbf{S}| = q^2+q+1-t$ it follows that on a point of $\mathcal{K}$ there are t lines of length q and $q+1-t$ lines of length $q+1$. Thus, $\mathbf{S}$ is a $\{0, 1, t\}$–semiaffine linear space with constant point degree, and so by Theorem 1.9 it is obtained from a projective plane of order q by deleting a set X of type $\{0, 1, t\}$, hence $|\mathbf{S}| = q^2+q+1-t$, where $t = |X|$. Each line through a point of $\mathcal{K}$ meets X either in 0 or 1 point, and every line on a point p not in $\mathcal{K}$ meets X in 0 or t points, and there is exactly one line through p meeting $\mathcal{K}$ in t points.

Therefore, all the points outside $\mathcal{K}$ are collinear, and $\mathbf{S}$ is obtained from a projective plane of order q by deleting t points on the line whose complement is $\mathcal{K}$. □

2.2. **Basic equations for a k–set of type (m, n).** Let $b = q^2+q+1+z$ be the number of lines of $\mathbf{S}$. Double counting gives:

$$t_m + t_n = b \tag{2.3}$$

$$mt_m + nt_n = k(q+1) \tag{2.4}$$

$$m(m-1)t_m + n(n-1)t_n = k(k-1) \tag{2.5}$$

From the first two equations it follows that

$$t_m = \frac{nb - k(q+1)}{n-m} \text{ and } t_n = \frac{k(q+1) - mb}{n-m} \tag{2.6}$$

Theorem 2.5. *If $\mathbf{S}$ admits a k–set $\mathcal{K}$ of type (m, n), then*

$$k^2 - k[1 + (m+n-1)(q+1)] + mnb = 0, \tag{2.7}$$

$$mq + n \leq k \leq (n-1)q + m \quad (0 < m < n < q+1), \tag{2.8}$$

$$(n-m) \mid q, \tag{2.9}$$

$$(n-m) \mid k-m. \tag{2.10}$$

Proof. Equation (2.7) is obtained by substituting (2.6) in (2.5). On a point Q not in $\mathcal{K}$ there is a constant number, say u_m, of m–secants of $\mathcal{K}$, and a constant number, say u_n of n–secants of $\mathcal{K}$.

Since $u_m + u_n = q + 1$, and $mu_m + nu_n = k$, it follows that

$$u_m = \frac{n(q+1) - k}{n - m}, \quad u_n = \frac{k - m(q+1)}{n - m}. \tag{2.11}$$

Also, on a point $p \in \mathcal{K}$ there is a constant number, say r_m, of m–secants of $\mathcal{K}$, and a constant number, say r_n of n–secants of $\mathcal{K}$, and so

$$r_m + r_n = q + 1, \quad (m-1)r_m + (n-1)r_n = k - 1.$$

It follows that

$$r_m = u_m - \frac{q}{n - m}, \quad r_n = u_n + \frac{q}{n - m}. \tag{2.12}$$

Thus[3], $n - m \mid q$, and from Equation (2.11) it follows that $n - m \mid k - m$.

Finally, we prove (2.8). In view of the previous section, we may assume that $m > 0$. If $n = q+1$, then on a point outside $\mathcal{K}$ there are only m–secants, and so $k = (q+1)m \le (q+1)(n-1) = (n-1)q - (q+1) < (n-1)q + m$. Now, assume that $n < q + 1$.

Let ℓ be an m–secant, and $p \in \mathcal{K} \cap \ell$. Each of the q lines through p, different from ℓ, meets $\mathcal{K} \setminus \{p\}$ in at most $n-1$ points, so $k \le (n-1)q + m$. Let ℓ' be an n–secant of $\mathcal{K}$, and $q \in \ell' \setminus \mathcal{K}$. The q–lines through q, different from ℓ', intersects $\mathcal{K}$ in at least m points, and so $k \ge mq + n$. □

2.3. k–sets of type (m, n).

Proposition 2.6. *Let $\mathcal{K}$ be a k–set of type $(m, q+1)$ in $\mathbf{S}$. Then, $\mathcal{K}$ contains at least one line of length $q+1$ of $\mathbf{S}$. Moreover, either $m = 1$ and $\mathcal{K}$ is a line of length $q+1$ of $\mathbf{S}$, or $m \ge 2$ and $q \le 2m - 2$.*

Proof. A line intersecting $\mathcal{K}$ in $q+1$ points has length $q+1$, so $q+1-h = q+1$ and $\mathcal{K}$ contains at least one line of length $q+1$. Equation (2.9) implies that $(q+1-m) \mid q$. Thus, either $m = 1$ or $q \le 2m-2$.

If $m = 1$, either $\mathcal{K}$ is a line of length $q+1$, or $\mathbf{S}$ is a projective plane and $\mathcal{K}$ coincides with $\mathbf{S}$. The second possibility cannot occur, since $\mathcal{K}$ is a two–character set. □

Proposition 2.7. *Let $\mathcal{K}$ be a k–set of type (m, n). Then $n - m = q$ if and only if either $\mathcal{K}$ is a line of length $q+1$, or $\mathbf{S}$ is obtained from a projective plane of order q by deleting t collinear points, $t \ge 0$, and $\mathcal{K}$ is the complement of the remaining $q + 1 - t$ points.*

Proof. $n - m = q \iff m + q = n \le q + 1$, and so either $m = 0$ and $n = q$ or $m = 1$ and $n = q+1$. By Theorem 2.4 $\mathcal{K}$ is of type $(0, q)$ if and only if $\mathbf{S}$ is obtained from a projective plane by deleting q by deleting t collinear points, $t \ge 0$, and $\mathcal{K}$ is the complement of the remaining $q + 1 - t$ points.

[3]For $m = 0$ we find the result of the previous section.

If $\mathcal{K}$ is of type $(1, q+1)$ from Proposition 2.6 it follows that $\mathcal{K}$ is a line of length $q+1$ of $\mathbf{S}$. $\square$

Proposition 2.8. *Let $\mathcal{K}$ be a k–set of type (m, n), then $n - m = 1$ if and only if either $\mathcal{K}$ is a point, or $\mathbf{S}$ is a projective plane of order q and $\mathcal{K}$ is the complement of a point of $\mathbf{S}$.*

Proof. If $m = 0$ or $n = q+1$, then either $\mathcal{K}$ is of type $(0, 1)$ or $(q, q+1)$. In the former case, by Proposition 2.1, $\mathcal{K}$ is a point. In the latter case, from Proposition 2.2, it follows that $\mathbf{S}$ is a projective plane of order q and $\mathcal{K}$ is the complement of a point in $\mathbf{S}$.

If $m > 0$ and $n < q+1$, using $n = m+1$ and (2.8) gives

$$mq + m + 1 = mq + n \leq k \leq (n-1)q + m = mq + m,$$

a contradiction. $\square$

3. k–sets of type $(1, n)$

Lemma 3.1. *Let q be a square, and let $\mathcal{K}$ be a k–set of type $(1, \sqrt{q}+1)$ in $\mathbf{S}$, with $k = q + \sqrt{q} + 1$. Then on every point of $\mathcal{K}$ there pass exactly $\sqrt{q}+1$ lines, and the geometry whose points are the points of $\mathcal{K}$ and whose lines are the intersections of $\mathcal{K}$ with the $(\sqrt{q}+1)$–secants of $\mathbf{S}$ is a projective plane of order $\sqrt{q}$.*

Proof. Let P be a point of $\mathcal{K}$, and let α denote the number of $(\sqrt{q}+1)$–secants of $\mathcal{K}$ passing through P. Then, $k = 1 + \alpha\sqrt{q}$. From $k = q + \sqrt{q} + 1$ it follows that $\alpha = \sqrt{q} + 1$.

Therefore, the pair $(\mathcal{K}, \mathcal{L}(\mathcal{K}))$, where

$$\mathcal{L}(\mathcal{K}) = \{\ell \cap \mathcal{K} \mid \ell \in \mathbf{S} \text{ and } \mid \ell \cap \mathcal{K} \mid = \sqrt{q} + 1\},$$

is a projective plane of order $\sqrt{q}$. $\square$

Lemma 3.2. *Let $\mathbf{S}$ be a finite linear space with constant point degree $q+1$, q square, and with constant line size. If there is a $(q + \sqrt{q} + 1)$–set $\mathcal{K}$ of type $(1, \sqrt{q}+1)$ in $\mathbf{S}$, then $\mathbf{S}$ is a projective plane of order q and $\mathcal{K}$ is a Baer subplane of $\mathbf{S}$.*

Proof. It follows from the previous lemma that on every point outside $\mathcal{K}$ there is exactly one line intersecting $\mathcal{K}$ in $\sqrt{q}+1$ points. Thus, the number of lines meeting $\mathcal{K}$ in exactly $\sqrt{q}+1$ points is $k = q + \sqrt{q} + 1$. Counting v via the lines through a point P gives $v = (q+1)(q-h)+1$. Double counting gives

$$\begin{aligned} v &= k + (q + 1 - h - \sqrt{q} - 1)(q + \sqrt{q} + 1) \\ &= (q + 1 - h - \sqrt{q})(q + \sqrt{q} + 1). \end{aligned}$$

Comparing the two values of v, one obtains $h = 0$, and the assertion easily follows. $\square$

Theorem 3.3. *Let* $\mathbf{S}$ *be a finite linear space with constant point degree* $q+1$*, and let* $\mathcal{K}$ *be a* $(q+n)$*–set of type* $(1,n)$ *of* $\mathbf{S}$*, with*[4] $q \leq (n-1)^2$*. Then* q *is a square and* $n = \sqrt{q}+1$*. Furthermore, if the lines of* $\mathbf{S}$ *have constant length, then* $\mathbf{S}$ *is a projective plane of order* q*, and* $\mathcal{K}$ *is a Baer subplane of* $\mathbf{S}$*.*

Proof. Let α denote the number of lines though a point of $\mathcal{K}$ which intersect $\mathcal{K}$ in n points. Then, $1+\alpha(n-1) = q+n$. So, $\alpha = q/(n-1)+1$. But α is the constant degree of the points of the linear space $(\mathcal{K}, \mathcal{L}(\mathcal{K}))$, and so $\alpha \geq n$. Thus, $q/(n-1)+1 \geq n$, that is $q \geq (n-1)^2$.

By our assumptions, $q \leq (n-1)^2$, and so $q = (n-1)^2$. If all the lines of $\mathbf{S}$ have the same length, the assertion follows from Lemma 3.2. □

Lemma 3.4. *Let* $\mathbf{S}$ *be a finite linear space with constant point degree* $q+1$ *and* $b = q^2+q+1+z$ *lines, and let* $\mathcal{K}$ *be a set of type* $(1,n)$ *of maximal size* $(n-1)q+1$ *in* $\mathbf{S}$*. Then,* $q \leq (n-1)^2$ *if and only if* $z \geq 0$*. In particular,* $q = (n-1)^2$ *if and only if* $z = 0$*.*

Proof. Through every point of $\mathcal{K}$ there pass a constant number α of lines intersecting $\mathcal{K}$ in n points. Thus, $k = \alpha(n-1)+1$. Since $k = (n-1)q+1$, it follows that $\alpha = q$. Therefore, on every point of $\mathcal{K}$ there is a single line intersecting $\mathcal{K}$ in exactly one point. So, $t_1 = k = (n-1)q+1$. Double counting gives $t_n n = kq$.

From $nt_1 + nt_n = nb$ it follows that

$$n(n-1)q + n + (n-1)q^2 + q = (q^2+q+1)n + zn,$$

that is,

$$q((n-1)^2 - q) = zn, \tag{3.1}$$

and so the assertion easily follows. □

Corollary 3.5. *Let* $\mathbf{S}$ *be a finite linear space with constant point degree* $q+1$ *and* $b \leq q^2+q+1+z$ *lines, and containing a set of type* $(1,n)$ *of maximal size* $(n-1)q+1$*. Then* $(n-1) \mid z$*.*

Proof. By (3.1) $q \mid zn$. Since, by Equation (2.9), $n-1 \mid q$, it follows that $(n-1) \mid zn$, and so $(n-1) \mid z$. □

Corollary 3.6. *An affine plane of order* q *does not contain a set of type* $(1,n)$ *of size* $(n-1)q+1$*.*

Theorem 3.7. *Let* $\mathbf{S}$ *be a finite linear space with constant point degree* $q+1$ *and with* $b = q^2+q+1$ *lines, and let* $\mathcal{K}$ *be a set of type* $(1,n)$ *of maximal size* $(n-1)q+1$ *in* $\mathbf{S}$*. Then* $q = (n-1)^2$*. Moreover, if all the lines have constant length, then* $\mathbf{S}$ *is a projective plane of order* q *and* $\mathcal{K}$ *is a unital of* $\mathbf{S}$*.*

[4]Note that if $z = 0$, Bruen's lower bound for the size of a blocking set of a projective plane also holds for finite linear spaces with constant point degree $q+1$, and so for $z = 0$, the assumption $q \leq (n-1)^2$ is not necessary.

Proof. Since $z = 0$, by Lemma 3.4 it follows that

$$q = (n-1)^2.$$

Thus $n = \sqrt{q} + 1$, and so $|\mathcal{K}| = q\sqrt{q} + 1$.

Now, assume that all the lines have constant length $q + 1 - h$. Then $v = |\mathbf{S}| = (q+1)(q-h) + 1$. On a point of $\mathcal{K}$ there is exactly one tangent. On a point outside $\mathcal{K}$ there are $1 + \sqrt{q}$ tangents. Counting, in two ways, the incident point–line pairs (x, ℓ), with $x \notin \mathcal{K}$, $x \in \ell$ and $|\ell \cap \mathcal{K}| = 1$, gives

$$(v - 1 - q\sqrt{q})(1 + \sqrt{q}) = (q\sqrt{q} + 1)(q - h),$$

from which it follows that

$$(q + \sqrt{q})(q - h) = q\sqrt{q}(1 + \sqrt{q}) = q(q + \sqrt{q}),$$

that is $h = 0$. Thus, $v = b = q^2 + q + 1$, $\mathbf{S}$ is a projective plane of order q, and $\mathcal{K}$ is a unital of $\mathbf{S}$. □

4. Two–character sets in a planar space with planes pairwise intersecting in the empty–set or in a line

Finite planar spaces whose planes pairwise intersect either in the empty–set or in a line, have their *local parameters* (that is the point–degree, the point–degree in every plane, the number of planes through a point, and the number of planes through a line) equal to those of the desarguesian projective space of dimension 3. It is an outstanding conjecture [5] to prove that they are obtained from $PG(3, q)$ by deleting a subset of points, in particular that their structure is between that of affine and projective space.

In particular, given such a finite planar space there is an integer $q \geq 2$ such that

- every point has degree $q^2 + q + 1$,
- through every point there are $q^2 + q + 1$ planes,
- in every plane each point has degree $q + 1$,
- through every line there are $q + 1$ planes,
- every plane contains at most $q^2 + q + 1$ lines,
- every plane contains at most $q^2 + q + 1$ points, and
- the number of points is at most $q^3 + q^2 + q + 1$.

Moreover, if there are no disjoint planes, the number of planes is $q^3 + q^2 + q + 1$ and every plane has $q^2 + q + 1$ lines. Thus, every plane of a finite planar space whose planes pairwise intersect in the empty–set or in a line is a linear space with constant point–degree $q + 1$.

Recall that every plane of such planar space is embeddable in a finite projective plane, actually if π is a plane and p is a point not in π, the plane π can be embedded, by projection through p, in the projective plane whose points are the lines through p and whose lines are the planes through p.

In this section we extend some result of Tallini and Tallini Scafati to these planar spaces, and we also give an embedding result. Throughout, we use the notation $(m, n)_1$ to denote a set of type (m, n) with respect to the lines of a linear space of dimension $d \geq 3$. We also recall, that Tallini [12] proves that a finite planar spaces with constant line size k and containing a set H of type $(0, k-1)_1$ is $\mathrm{P}G(d, q)$ and H is the complement of a prime[5] in $\mathrm{P}G(d, q)$.

Proposition 4.1. $\mathbf{S}$ *has no k–sets of type* $(1, q)_1$.

Proof. First, assume $q > 2$. Let $\mathcal{K}$ be a set of type $(1, q)_1$ in $\mathbf{S}$. Let $p \in \mathcal{K}$ and $r \notin \mathcal{K}$, every plane passing through the line pr intersects $\mathcal{K}$ in a k'–set of type $(1, q)_1$. Since $q - 1 \geq 2$ does not divide q, from (2.9) the assertion follows. The case $q = 2$ cannot occur by (2.8). □

Theorem 4.2. *Let* $\mathbf{S}$ *be a planar space whose planes pairwise intersect either in the empty set or in a line, and let* $\mathcal{K}$ *be a set of type* $(0, n)_1$ *in* $\mathbf{S}$*, then* $n = 1$ *or* $n = q$*. If* $n = 1$*,* $\mathcal{K}$ *is a point. If* $n = q$*, there are lines of length* $q + 1$*,* $\mathbf{S}$ *is obtained from* $\mathrm{P}G(3, q)$ *by deleting a subset* X *of points, and* $\mathcal{K}$ *is the complement of a (possibly punctured) plane* π *of* $\mathbf{S}$.

Proof. Let ℓ be a line missing $\mathcal{K}$, by Proposition 1.1 every plane α through ℓ and containing a point of $\mathcal{K}$ intersects $\mathcal{K}$ in a set $\mathcal{K}'$ of type $(0, n)_1$ in α, and so $|\mathcal{K}'| = (q+1)(n-1)+1$. If x is the number of planes through ℓ intersecting $\mathcal{K}$, then $k = |\mathcal{K}| = x[(q+1)(n-1)+1]$.

Let p be a point of $\mathcal{K}$, each line through p is n–secant, so $k = 1 + (q^2 + q + 1)(n-1)$. Comparing the two values of k, one obtains

$$x = q - \frac{q-n}{(q+1)(n-1)+1},$$

thus $(q+1)(n-1)+1 | (q-n)$. Let $n \neq q$. From $(q+1)(n-1)+1 \leq (q-n)$ it follows that $n = 1$, otherwise $q + 2 \leq q - n < q$. Clearly, for $n = 1$ $\mathcal{K}$ is a point. So, assume $n = q$. Each line through p has at least q points, hence $v \geq (q^2+q+1)(q-1)+1 = q^3$. By a theorem of Metsch [7], $\mathbf{S}$ is embeddable in a 3–dimensional projective space.

If every line has at most q points, then $\mathcal{K}$ is a subspace of $\mathbf{S}$, and since $k = v$ it follows that $\mathcal{K}$ is the whole space, a contradiction, since $\mathcal{K}$ has external lines. Thus, there is at least one line of length $q + 1$.

Let x be a point outside $\mathcal{K}$, the number of lines passing through x and secant $\mathcal{K}$ are $k/q = q^2$. Thus, through x there are $q + 1$ line external to $\mathcal{K}$, and q^2 lines of length $q + 1$. The lines through x and external to $\mathcal{K}$ span a (possibly punctured) plane π disjoint from $\mathcal{K}$, otherwise there is a plane spanned by two of such lines which intersects $\mathcal{K}$ in a set of type $(0, q)$, that is, in the complement of a line (see Theorem 2.4), a contradiction.

Therefore, $\mathcal{K}$ is the complement of π in $\mathbf{S}$. □

Corollary 4.3. $\mathrm{A}G(r, q)$*,* $r \geq 3$*, has no set of type* $(0, n)_1$*, if* $n \geq 2$.

[5]In a planar space a *prime* is a subspace S which meets every line not contained in S in exactly one point.

Proof. By contradiction, assume that $\mathcal{K}$ is a set of type $(0,n)_1$ in $\mathrm{A}G(r,q)$. Let $\mathrm{A}G(3,q)$ be a 3–dimensional subspace of $\mathrm{A}G(r,q)$ intersecting $\mathcal{K}$ in at least two points, then $\mathrm{A}G(3,q)\cap\mathcal{K}$ is a set of type $(0,n)_1$ in a planar space whose planes are as in the assumptions of the previous theorem, and so $\mathrm{A}G(3,q)$ has lines of length $q+1$, a contradiction. □

Corollary 4.4 (Tallini Scafati [9]). *If* $\mathrm{P}G(r,q)$*,* $r\geq 3$*, has a set of type* $(0,n)_1$*, then* $n=1$ *or* $n=q$ *and either* $\mathcal{K}$ *is a point or the complement of a hyperplane.*

Proof. Assume $\mathcal{K}$ is not a point, and let $H=\mathrm{P}G(3,q)$ be a subspace of dimension 3 meeting $\mathcal{K}$, in at least two points. By Proposition 1.1 the set $\mathcal{K}'=H\cap\mathcal{K}$ is of type $(0,n)_1$ in H, and so from Theorem 4.2 and since $\mathcal{K}$ is not a point, it follows that $n=q$. Now, the assertion easily follows. □

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