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THE FLOW LATTICE OF ORIENTED MATROIDS

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ABSTRACT. Recently Hochstättler and Nešetřil introduced the flow lattice of an oriented matroid as generalization of the lattice of all integer flows of a digraph or more general a regular matroid. This lattice is defined as the integer hull of the characteristic vectors of signed circuits.

Here, we characterize the flow lattice of oriented matroids that are uniform or have rank 3 with a particular focus on the dimension of the lattice and construct a basis consisting of directed circuits. For general oriented matroids we introduce a 2-sum and decompose oriented matroids into 3-connected parts. We show how to determine the dimension of the lattice of 2-sums and conclude with some questions based on extensive experiments on small oriented matroids with connectivity at least 3.

1. INTRODUCTION

A circular flow in a digraph $G = (V, A)$ is a flow $f \in \mathbb{Z}^{|A|}$ which satisfies Kirchhoff's law, *i.e.* the in-flow of a vertex equals the out-flow:

$$\sum_{a \in \delta^+(v)} f_a = \sum_{a \in \delta^-(v)} f_a \qquad \forall v \in V.$$

It is well known that each circular flow must be an integer combination of characteristic vectors of signed circuits of G and obviously each integer combination of signed circuits is a circular flow as well (see [1]).

Thus, the *lattice of integer flows* is generated by the oriented circuits and this easily generalizes to oriented matroids. Analogous to the flow number of a digraph (see [19] for a survey), [14] introduced the flow number $\Phi_{\mathcal{L}}(\mathcal{O})$ of an oriented matroid as the smallest number k for which a nowhere-zero k-flow exists. For co-graphic oriented matroids this flow number corresponds to the chromatic number of the corresponding graph.

In this work we characterize the *flow lattice* of uniform and rank 3 oriented matroids by computing its dimension, presenting an easy to test membership condition and constructing a basis of signed circuits. Our observations are summarized in Table 1.

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From this structure the flow number $\Phi_{\mathcal{L}}$ for the two classes of oriented matroids is easily derived. While $\Phi_{\mathcal{L}}$ is invariant for all orientations of a uniform orientable matroid, the dimension of the flow lattice turns out to be not. The dimension is either n or $n-1$ (where n is the number of elements) and dimension $n-1$ is achieved by the class of realizable oriented matroids which represent neighborly polytopes (see [10] for an introduction and [13] and [21] for lectures on neighborly polytopes). Generalizing to non-realizable matroids, exactly the reorientation classes of neighborly matroid polytopes come with a flow lattice dimension of $n-1$. Note, that not every neighborly matroid polytope is realizable ([2], [5]).

For uniform or rank 3 oriented matroids the flow number of [14] does not depend on the orientation of the underlying matroid, however, different orientations of the same matroid may lead to different flow lattice dimensions. This might indicate that $\Phi_{\mathcal{L}}$ is not a matroid invariant (this problem is raised in [14]), yet we have not found orientations of a matroid where the numbers differ.

Our results about the dimension of the flow lattice solve a research problem raised in [3, Exercise 4.45(d)*] for rank 3 and uniform matroids. For general oriented matroids we give an outlook based on the decomposition into sums and 2-sums of 3-connected oriented matroids and compute the dimension of the flow lattice of 2-sums from the dimension of the components. Furthermore we report on computational experiments on the data of [8] and standard examples from [17]. Our numerical results – on examples that still have to be considered as very small – seem to indicate that the flow lattice dimension of non-regular non-uniform 3-connected oriented matroids might be in $\{n-1, n\}$ except for a small family of examples and furthermore, for a growing number of elements and fixed rank the flow lattice seems to become trivial.

The latter matches the results for a different approach to define a flow number Φ_o of an oriented matroid (see [12]). This flow number is not a matroid invariant even for uniform orientable matroids ([11]) and, in the non-regular case, seems not to be related to the parameter considered here.

2. Notation

2.1. **Oriented Matroids.** We use standard notation for oriented matroids as in [3]. Let $E = \{1, \ldots, n\}$ be an index set and $\mathcal{C}$ the set of *signed circuits* of an oriented matroid $\mathcal{O}(E, \mathcal{C})$. For a signed subset $C = (C^+, C^-)$ we denote by $\vec{C}$ its corresponding signed vector and by $\vec{C}_1 \circ \vec{C}_2$ the signed vector of the *composition* of C_1 and C_2 (see [3] for details). $A(\mathcal{C}) \in \{0, \pm 1\}^{|\mathcal{C}| \times n}$ denotes the *circuit matrix* holding the signed vectors of circuits as rows (occasionally we skip one of two alternating rows). We call a circuit $C \in \mathcal{C}$ *balanced* if $|C^+| = |C^-|$. If this condition holds for any circuit $C \in \mathcal{C}$ then we call $\mathcal{O}(E, \mathcal{C})$ to be *balanced* (note that this definition is dual to that of

	non-regular			
	non-uniform	uniform		
	rank 3	r even	r odd	
$\dim \mathcal{F}_{\mathcal{O}}$	$\|E\|$	$\|E\|$	$\|E\| - 1$	$\|E\|$
$\mathcal{F}_{\mathcal{O}}$	$\mathbb{Z}^{\|E\|}$	$\mathbb{Z}^{\|E\|}$	$\{v\}^{\perp} \cap \mathbb{Z}^{\|E\|}$ $v \in \{1, -1\}^{\|E\|}$	$\{x^T \mathbb{1} \text{ even}\}$
$\Phi_{\mathcal{L}}$	2	2	2, if $\|E\|$ even 3, if $\|E\|$ odd	2, if $\|E\|$ even 3, if $\|E\|$ odd

TABLE 1. Results of the paper concerning the flow lattice of a connected simple and co-simple oriented matroid $\mathcal{O}$ with more than $r + 3$ elements

[20] where a balanced oriented matroid of odd rank has balanced co-circuits only).

2.2. **Matroid Polytopes.** Consider an acyclic oriented matroid $\mathcal{O}$ (*i.e.* for any circuit C, $|C^-| > 0$). Then the complements of the positive co-circuits of $\mathcal{O}$ are said to be the *facets* of $\mathcal{O}$ and any intersection of facets is a *face* of $\mathcal{O}$. We say that $\mathcal{O}$ is a *matroid polytope* if all one-element subsets of E are faces of $\mathcal{O}$.

Definition 2.1. *A matroid polytope $\mathcal{O}$ of rank r on the ground set $E = \{1, \ldots, n\}$ is called* neighborly *if any subset $F \subseteq E$, $|F| \leq \lfloor (r-1)/2 \rfloor$ is a face of $\mathcal{O}$. An oriented matroid is called* neighborly *if it is reorientation equivalent to a neighborly matroid polytope.*

2.3. **Flows in Oriented Matroids.** Hochstättler and Nešetřil introduced the flow lattice of an oriented matroid as the integer hull of its signed vectors:

Definition 2.2. *For $V = \{v_1, \ldots, v_m\} \subset \mathbb{Z}^n$ let*

$$\operatorname{lat}(V) = \left\{ \sum_{i=1}^{m} \lambda_i v_i : \lambda_i \in \mathbb{Z} \right\}$$

denote the integer lattice of V. The flow lattice *of an oriented matroid $\mathcal{O}(E, \mathcal{C})$ and its orthogonal complement are denoted by*

$$\mathcal{F}_{\mathcal{O}} := \operatorname{lat}(\{\vec{C} : C \in \mathcal{C}\})$$
$$\mathcal{F}_{\mathcal{O}}^{\perp} := \{x \in \mathbb{R}^n : x^T y = 0 \text{ for all } y \in \mathcal{F}_{\mathcal{O}}\}.$$

The flow number *$\Phi_{\mathcal{L}}(\mathcal{O})$ of an oriented matroid $\mathcal{O}$ is defined as the smallest $k \in \mathbb{N}$ such that there is an $x \in \mathcal{F}_{\mathcal{O}}$ which satisfies $0 < |x_e| < k$ for all $e \in E$.*

This definition of a flow number coincides with the standard definition in case $\mathcal{O}$ is regular. [14] could show that the flow number is a matroid invariant for uniform orientable matroids. Generalizing to non-uniform orientable matroids they asked:

Problem 2.3. *Is there an orientable matroid $\mathcal{M}$ so that $\mathcal{M}$ has signings $\mathcal{O}_1, \mathcal{O}_2$ satisfying $\Phi_{\mathcal{L}}(\mathcal{O}_1) \neq \Phi_{\mathcal{L}}(\mathcal{O}_2)$?*

As obviously, the flow lattices of two oriented matroids from the same reorientation class must be isomorphic, we consider how far this might extend to different reorientation classes. We will show that they even need not have the same dimension.

3. Uniform Oriented Matroids

[14] have shown that for a uniform oriented matroid with even rank r the flow lattice is *trivial*, *i.e.* $\mathcal{F}_{\mathcal{O}} = \mathbb{Z}^n$. We will now deal with the case of odd rank.

Theorem 3.1. *Let $\mathcal{O}$ be a uniform oriented matroid on $E = \{1, \ldots, n\}$ of odd rank $r \leq n-2$ and $\dim \mathcal{F}_{\mathcal{O}} < n$. Then, there is a reorientation ${}_I\mathcal{O}$ of $\mathcal{O}$ such that $\mathcal{F}^{\perp}_{{}_I\mathcal{O}} = \mathbb{1} \cdot \mathbb{R}$. In particular $\dim \mathcal{F}_{\mathcal{O}} = n-1$.*

Proof. We proceed by induction on $n \geq r+2$.

Let therefore $n = r+2$. Then, the dual $\mathcal{O}^*$ is the n-point line where each point corresponds to a co-circuit of $\mathcal{O}^*$ (resp. a circuit of $\mathcal{O}$). An alternating orientation of the points along the line yields the circuit matrix

$$A(\mathcal{C}) = \begin{pmatrix} 0 & +1 & -1 & +1 & -1 & \cdots \\ +1 & 0 & -1 & +1 & -1 & \cdots \\ +1 & -1 & 0 & +1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

of $\mathcal{O}$. It can be verified that $A(\mathcal{C})$ has rank $n-1$ and, since circuits have even size $r+1$, $\mathbb{1}$ is orthogonal to any signed circuit.

Now let $n > r+2$ and $\mathcal{O}$ a uniform oriented matroid on n elements such that $\dim \mathcal{F}_{\mathcal{O}} < n$. We choose $x \in \mathcal{F}^{\perp}_{\mathcal{O}} \backslash \{0\}$ and assume that $\mathcal{O}$ is oriented such that $x \geq 0$. Clearly, for an arbitrary $e \in E$, $x \backslash e$ is orthogonal to all signed circuits of $\mathcal{O} \backslash e$. Thus, by induction $x_f = x_g$ for all $f, g \in E \backslash e$. Since e was arbitrarily chosen and $|E| \geq 3$ we can conclude $x \in \mathbb{1} \cdot \mathbb{R}$. □

Corollary 3.2. *For the flow lattice of ${}_I\mathcal{O}$ as in Theorem 3.1 we have $\mathcal{F}_{{}_I\mathcal{O}} = \{\mathbb{1}\}^{\perp} \cap \mathbb{Z}^n$.*

Proof. By Theorem 3.1 the inclusion $\mathcal{F}_{{}_I\mathcal{O}} \subseteq \{\mathbb{1}\}^{\perp} \cap \mathbb{Z}^n$ holds. Now let $y \in \{\mathbb{1}\}^{\perp} \cap \mathbb{Z}^n$ be arbitrary. As the circuit matrix of $\mathcal{O}$ has even rank and thus $\mathcal{F}_{\mathcal{O}/n}$ has full dimension we find signed circuits $D_1, \ldots, D_k \in \mathcal{O}/n$ and integers $\lambda_1, \ldots, \lambda_k$ such that $y \setminus n = \sum_{i=1}^{k} \lambda_i \vec{D}_i$. Now let C_i denote the signed circuit of $\mathcal{O}$ satisfying $C_i \setminus n = D_i$ then necessarily we must have $y = \sum_{i=1}^{k} \lambda_i \vec{C}_i$. □

Note, that in the reorientation classes considered there is a representative which is balanced. These oriented matroids are exactly neighborly matroid polytopes of odd rank, which is an easy consequence using the following proposition which seems to be folklore.

Proposition 3.3 (see [3, Remark 9.4.10]). *A matroid polytope $\mathcal{O}$ of rank $2k+1$ is neighborly if and only if $|C^+| = |C^-| = k+1$ for all circuits $C \in \mathcal{O}$.*

Theorem 3.4. *Let $\mathcal{O}$ be a uniform oriented matroid of odd rank. Then $\dim \mathcal{F}_{\mathcal{O}} \leq n-1$ if and only if $\mathcal{O}$ is neighborly.*

Proof. By Theorem 3.1 $\dim \mathcal{F}_{\mathcal{O}} \leq n-1$ if and only if there is a reorientation I such that $\mathcal{F}^{\perp}_{{}_I\mathcal{O}} = \{\mathbb{1}\} \cdot \mathbb{R}$. This reorientation makes $\mathcal{O}$ into a balanced matroid polytope which is neighborly by Proposition 3.3. □

Next, we analyze the case of $\mathcal{F}_{\mathcal{O}}$ for odd rank and full dimension. For an inductive proof we need to avoid that $\mathcal{O}$ becomes neighborly after deleting an element. The next lemma guarantees the existence of such an element.

Lemma 3.5. *Let $\mathcal{O}$ be a uniform oriented matroid of odd rank r on $E = \{1, \dots, n\}$, where $n > r+3$. If $\mathcal{O} \setminus i$ is neighborly for all $1 \leq i \leq n$ then $\mathcal{O}$ is neighborly, too.*

Proof. Let $\mathcal{O}$ be oriented such that $\mathcal{O} \setminus n$ has balanced circuits only. It suffices to show that, eventually reorienting n, all circuits in $\mathcal{O}$ are balanced. Assume, there is an unbalanced circuit C. As all circuits have size $r+1$ and $n > r+3$, eventually relabeling the elements, we may assume that $n-1 \notin C$. Now, let $I \subseteq \{1, \dots, n-2, n\}$ such that ${}_I\mathcal{O} \setminus (n-1)$ has balanced circuits only. Then $I \cap \tilde{C}$ has to be even for any circuit $\tilde{C} \subseteq E \setminus \{n-1, n\}$, *i.e.* for all $r+1$ subsets of $\{1, \dots, n-2\}$. As $n-2 > r+1$ this implies $I = \{n\}$ or $I = \{1, \dots, n-2\}$. We will show that there can be no other unbalanced circuit $\hat{C} \supseteq \{n-1, n\}$. Eventually relabeling the elements we may assume that $n-2 \notin \hat{C}$. Now, repeating the above argument for n and $n-1$ we find an $\hat{I}$ such that ${}_{\hat{I}}\mathcal{O} \setminus (n-2)$ has balanced circuits only and necessarily $\hat{I} \in \{\{n\}, \{1, \dots, n-3, n-1\}\}$ as well as $\hat{I} \in \{\{n-1\}, \{1, \dots, n-3, n\}\}$, which is a contradiction. □

Proposition 3.6. *The following subsets of $\mathbb{Z}^n$ are equal:*

(1) $N_1 = \{x \in \mathbb{Z}^n : \mathbb{1}^T x \text{ is even}\}$
(2) $N_2 = \operatorname{lat}(\{e_i \pm e_j : i \neq j \in E\})$
(3) $N_3 = \operatorname{lat}(\{e_{i_0} - e_j : j \in E, j \neq i_0\} \cup \{v\})$ *for* $v \in \{2e_{j_0}, e_{j_0} + e_{k_0}\}$ *and fixed* $i_0, j_0, k_0 \in E$

We now can characterize the full dimensional flow lattices of uniform oriented matroids of odd rank as follows:

Theorem 3.7. *Let $\mathcal{O}$ be a uniform oriented matroid on a finite set $E = \{1,\dots,n\}$ of odd rank r and $n \geq r+3$. If $\dim \mathcal{F}_{\mathcal{O}} = n$ and $x \in \mathbb{Z}^n$ then*

$$x \in \mathcal{F}_{\mathcal{O}} \iff \sum_{i=1}^{n} x_i \text{ is even.}$$

Proof. We proceed by induction on n. If $n = r+3$, then $\mathcal{O} \setminus e$ can be reoriented to be balanced for all $e \in E$. Let $\mathcal{O}$ be oriented such that $\mathcal{O} \setminus n$ is balanced. By Corollary 3.2 $\mathcal{F}_{\mathcal{O}\setminus n} = \mathbb{1}^{\perp} \cap \mathbb{Z}^{n-1}$. Hence, $e_i - e_j \in \mathcal{F}_{\mathcal{O}}$ for $\{i,j\} \subseteq \{1,\dots,n-1\}$. As $\mathcal{O}$ is imbalanced we may assume that $\mathcal{O} \setminus 1$ is imbalanced. Let $v \in \{1,-1\}^{n-1}$ be the vector such that $\mathcal{F}^{\perp}_{\mathcal{O}\setminus 1} = v \cdot \mathbb{R}$. As $\mathcal{O} \setminus 1$ is imbalanced there is an index j such that $v_2 = -v_j$. We may assume that the orientation of $\mathcal{O}$ has been chosen such that $v_2 = v_n$ and thus $j \neq n$. We conclude $\{e_2 + e_j, e_2 - e_n\} \subset \mathcal{F}_{\mathcal{O}}$. By Proposition 3.6 these vectors generate all integer vectors with even component sum. Finally, as each of the generators has even component sum the induction is founded.

Now assume $n > r+3$. By Lemma 3.5 there is $i \in E$ such that $\dim \mathcal{F}_{\mathcal{O}\setminus i} = n-1$. By induction we find that all integer vectors with even sum that are zero in i belong to $\mathcal{F}_{\mathcal{O}}$. Now, let C be a circuit containing i. From this we derive $e_i + e_j$ or $e_i - e_j$ in $\mathcal{F}_{\mathcal{O}}$ for some $j \in E$. We also get $2e_i$ by subtracting twice the other unit vectors from $2\vec{C}$ and the claim follows by Proposition 3.6. □

Summarizing we get from [14], Corollary 3.2, and Theorem 3.7:

Theorem 3.8. *Let $\mathcal{O}$ be a uniform oriented matroid on $n \geq r+2$ elements. Then*

$$\mathcal{F}_{\mathcal{O}} = \begin{cases} \mathbb{Z}^n, & \text{if } r \text{ is even,} \\ \{v\}^{\perp} \cap \mathbb{Z}^n \text{ for some } v \in \{1,-1\}^n, & \text{if } r \text{ is odd and } \mathcal{O} \text{ is neighborly,} \\ \{x \in \mathbb{Z}^n : x^T \mathbb{1} \text{ is even}\}, & \text{otherwise.} \end{cases}$$

Regarding the flow number of a uniform orientable matroid we get the result of [14] as an easy corollary.

Corollary 3.9.

$$\Phi_{\mathcal{L}}(\mathcal{O}) = \begin{cases} 2, & \text{if } nr \text{ is even,} \\ 3, & \text{if } nr \text{ is odd.} \end{cases}$$

4. Non-uniform Oriented Matroids with Rank 3

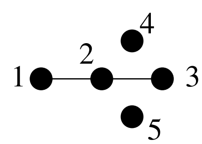
While in the last section $\mathcal{O}$ was assumed to be uniform we now turn to general oriented matroids but limit the rank to be 3. In particular, we will show that any rank-preserving non-regular simple and co-simple extension of a regular oriented matroid increases the dimension of the flow lattice by 4 (instead of at most two in the uniform case).

While in the uniform case we did not have to deal with loops, co-loops, parallels or co-parallels, we now have to consider these cases separately. The following proposition holds for arbitrary ranks.

Proposition 4.1. *Let $\mathcal{O}$ be an oriented matroid on the ground set E and $e \in E$. Then*

$$\dim \mathcal{F}_{\mathcal{O}} = \begin{cases} \dim \mathcal{F}(\mathcal{O} \setminus e) + 1, & \text{if } e \text{ is a loop,} \\ \dim \mathcal{F}(\mathcal{O} \setminus e) + 1, & \text{if } e \text{ is parallel to some } f \in E, \\ \dim \mathcal{F}(\mathcal{O}/e), & \text{if } e \text{ is a co-loop,} \\ \dim \mathcal{F}(\mathcal{O}/e), & \text{if } e \text{ is co-parallel to some } f \in E. \end{cases}$$

The oriented matroid in Figure 1 (which we call $\mathcal{O}_5$) is contained in almost any non-regular and non-uniform oriented matroid of rank 3.



$$\begin{matrix} + & - & + & 0 & 0 \\ + & - & 0 & + & + \\ + & 0 & - & + & + \\ 0 & + & - & + & + \end{matrix}$$

FIGURE 1. The oriented matroid $\mathcal{O}_5$ with the signs of its circuit matrix

$\mathcal{O}_5$, being a co-extension of the 4-point line by a co-parallel, has a flow lattice dimension of 4 and

$$\mathcal{F}(\mathcal{O}_5) = \operatorname{lat}(\{e_1, e_2, e_3, e_4 + e_5\}).$$

At first, we point out that the flow lattice is trivial for any $U_{2,4}$-free extension of $\mathcal{O}_5$.

Lemma 4.2. *Any connected single element extension of $\mathcal{O}_5$ which does not contain a 4-point line has trivial flow lattice.*

Proof. Consider a single element extension $\mathcal{O}_6$ of $\mathcal{O}_5$ which does not contain a 4-point line. Hence, $\{1, 2, 3, 6\}$ is not a hyperplane and $\{4\}$ and $\{5\}$ are no longer co-parallel. Consequently, there are circuits C_5 and C_6 in $\mathcal{O}_6$ that contain $\{4, 6\}$ resp. $\{5, 6\}$ and not both of 4 and 5. We show that the circuits of $\mathcal{O}_6$ generate $\{e_i\}_{i=1,\dots,6}$. Let $C_1, \dots, C_4$ be the signed circuits of $\mathcal{O}_5$ as shown in Figure 1. Then $e_1 = \vec{C}_1 - \vec{C}_2 + \vec{C}_3$, $e_2 = \vec{C}_1 - \vec{C}_2 + \vec{C}_4$ and $e_3 = \vec{C}_1 - \vec{C}_3 + \vec{C}_4$. Since any circuit of $\mathcal{O}_5$ is also a circuit in $\mathcal{O}_6$, we can skip the first three coordinates from our considerations (we will fill these coordinates with $*$).

Assume that there is no circuit containing 4, 5 and 6. Hence, $X_\tau := \{\tau, 4, 5, 6\}$ is a dependent set for any $\tau \in \{1, 2, 3\}$ but not a circuit. Due to the simplicity of $\mathcal{O}_6$ and the absence of co-parallel elements 6 must either extend the 3-point line of $\mathcal{O}_5$ which is excluded by assumption or $X_\tau \setminus \epsilon$ is a circuit for $\epsilon \in \{4, 5\} \setminus \tau$. In other words, there must be circuits $\{1, \epsilon_1, 6\}, \{2, \epsilon_2, 6\}, \{3, \epsilon_3, 6\}$ with $\epsilon_i \in \{4, 5\}$. Assume w.l.o.g. that $\epsilon_1 = \epsilon_2$. By circuit elimination we find a circuit contained in $\{1, 2, \epsilon_1\}$, a contradiction.

Hence, there must be a circuit C_7 containing $\{4,5,6\}$. We may choose a reorientation so that the rows of

$$\begin{array}{r} C_7 = \\ C_5 = \\ C_6 = \end{array} \begin{pmatrix} * & * & * & 1 & 1 & 1 \\ * & * & * & \alpha & 0 & 1 \\ * & * & * & 0 & \beta & 1 \end{pmatrix} =: (* \mid B)$$

correspond to circuits of $\mathcal{O}_6$ where $\alpha, \beta \in \{+1, -1\}$. If $(\alpha, \beta) \neq (-1, -1)$ the rows of B span $\mathbb{Z}^3$ with integer coefficients. Otherwise, applying circuit elimination between C_5 and C_6 we find a circuit $C \in \mathcal{C}(\mathcal{O}_6)$ containing 4 and not 6. Since 4 and 5 are co-parallel in $\mathcal{O}_5$ and C is also a circuit in $\mathcal{O}_5$ we necessarily must have $C(4) = C(5)$ contradicting signed circuit elimination which would yield $C(4) = -C(5)$. Consequently, any one-element extension of $\mathcal{O}_5$ has a trivial flow lattice. $\square$

By considering the enumeration of oriented matroids of [8] we checked that any 4-line-free connected single element extension of $\mathcal{O}_5$ is an orientation of one of P_6, R_6, Q_6 or $\mathcal{W}^3$ which, by the above lemma, has a trivial flow lattice. On the other hand, any orientation of P_6, R_6, Q_6 or $\mathcal{W}^3$ contains a reorientation of $\mathcal{O}_5$ as a minor.

Lemma 4.3. *Let $\mathcal{O}$ be a simple non-uniform oriented matroid on $n > 5$ elements which does not contain the deletion minors P_6, R_6, Q_6 or $\mathcal{W}^3$. Then $\mathcal{O}$ must be either an orientation of $\mathcal{M}(K_4)$ or contains an $(n-2)$-point line.*

Proof. Assume that $U_{2,n-2}$ is not a deletion minor and $\underline{\mathcal{O}} \ncong \mathcal{M}(K_4)$. Then we can find a deletion minor $\mathcal{O}_6$ of $\mathcal{O}$ with 6 elements which contains a 3-point line but not a 4-point line and is not isomorphic to $\mathcal{M}(K_4)$. Note, that only the non-orientable Fano plane has no other deletion minor than $\mathcal{M}(K_4)$ (see [17]). This matroid must have a non-uniform, non-regular deletion minor with 5 points. Since up to reorientation $\mathcal{O}_5$ is the unique non-regular, non-uniform oriented matroid on five elements without four point line, $\mathcal{O}_6$ must be an orientation of P_6, R_6, Q_6 or $\mathcal{W}^3$. $\square$

Since the lattice of an extension of an oriented matroid with trivial flow lattice again must be trivial we find that a simple and co-simple rank 3 oriented matroid with more than 5 elements has trivial flow lattice if it is not isomorphic to $\mathcal{M}(K_4)$. As a consequence, any simple and co-simple single element extension of an orientation of $\mathcal{M}(K_4)$ increases the dimension of $\mathcal{F}_{\mathcal{O}}$ by $r+1=4$ as $\dim \mathcal{F}_{\mathcal{O}} = n - r$ for regular oriented matroids.

Theorem 4.4. *Let $\mathcal{O}$ be a simple and co-simple non-uniform oriented matroid of rank 3 on a ground set E with $n \geq 6$ elements. Then $\mathcal{F}_{\mathcal{O}} = \mathbb{Z}^n$ if and only if $\underline{\mathcal{O}} \ncong \mathcal{M}(K_4)$.*

In the next section we will construct a basis of $\mathcal{F}_{\mathcal{O}}$ containing only signed circuits. Note, that in general a generating set of an integer lattice may not contain a basis.

5. Constructing a Basis of Circuits

Before we construct a basis for the flow lattice of a uniform oriented matroid we briefly discuss the rank 3 case. By Proposition 4.1 we may assume that $\mathcal{O}$ is co-simple. If $\mathcal{O}$ is regular the elementary circuits with respect to an arbitrary basis form a basis of the lattice as well. The uniform case is discussed below. Otherwise, $\mathcal{O}$ either is $\mathcal{O}_5$ or contains an orientation $\mathcal{O}_6$ of P_6, R_6, Q_6 or $\mathcal{W}^3$ as a deletion minor. Bases for these instances are easily constructed as in the proof of Lemma 4.2. Now, it suffices to add an arbitrary circuit in $\mathcal{O}_6 + e$ for each element in $\mathcal{O}$ not contained in $\mathcal{O}_6$.

In the uniform case we can usually start with a deletion minor of $r+2$ points and construct a basis inductively. The only case which has to be handled with some more care is when the dimension increases by 2. Lemma 3.5 guarantees that we can select an order of our elements such that this increase happens when we add the $(r+3)^{rd}$ point.

Theorem 5.1. *Let $\mathcal{O}$ be a uniform or rank 3 oriented matroid on $E = \{1, \dots, n\}$ of rank r. Then the flow lattice $\mathcal{F}_{\mathcal{O}}$ has a basis consisting of characteristic vectors of circuits.*

Proof. The non-uniform rank 3 case has been considered already. We proceed by induction on n. The assertion is trivial for $n \in \{r, r+1\}$.

Case 1: $n = r + 2$:

[14] have shown that for even rank $A(\mathcal{C})$ has full rank. As it contains an orientation of each circuit, they must form a basis of the flow lattice. If r is odd, then, by Theorem 3.1 $A(\mathcal{C})$ has rank $n-1$ and, by symmetry, there is a vector $v \in \{1, -1\}$ in the kernel of $A(\mathcal{C})^\top$. Thus, deleting an arbitrary circuit from $A(\mathcal{C})$ yields a basis.

Case 2: $n > r + 2$:

First, we consider the case that there exists an element $j \in \{1, \dots, n\}$ such that $\dim(\mathcal{F}_{\mathcal{O}}) = \dim(\mathcal{F}_{\mathcal{O}\setminus j}) + 1$. We may assume $j = n$. Let $C_1, \dots, C_k$ be such that their characteristic vectors form a basis of $\mathcal{F}_{\mathcal{O}\setminus n}$. Let C_{k+1} be an arbitrary circuit containing n. We have to distinguish the following three sub-cases:

Case 2.1: r is even:

As $\mathcal{F}_{\mathcal{O}\setminus n} = \mathbb{Z}^{n-1}$ we have $e_1, \dots, e_{n-1}$ as integer combinations of $\vec{C}_1, \dots, \vec{C}_k$ and, thus e_n as an integer combination of $\vec{C}_1, \dots, \vec{C}_k, \vec{C}_{k+1}$.

Case 2.2: r is odd, $\dim(\mathcal{F}_{\mathcal{O}\setminus n}) = n - 1$:

Here, $\vec{C}_1, \dots, \vec{C}_k$ by Theorem 3.7 generate all integer vectors with even component sum. Using this fact, it is immediate how to get $e_n \pm e_i$ and $2e_n$ as integer combinations of $\vec{C}_1, \dots, \vec{C}_k, \vec{C}_{k+1}$.

Case 2.3: r is odd, $\dim(\mathcal{F}_{\mathcal{O}\setminus n}) = n - 2 = \dim(\mathcal{F}_{\mathcal{O}}) - 1$:

We may assume that $\mathcal{O}$ is oriented such that $\mathcal{F} = \{\mathbb{1}\}^\perp \cap \mathbb{Z}^n$. Then C_{k+1} is balanced and by induction $e_i - e_j$ are integer combinations of $\vec{C}_1, \dots, \vec{C}_k$ for $\{i, j\} \subseteq \{1, \dots, n-1\}$. Subtracting suitable $(r-1)/2$

of these vectors from $\vec{C}_{k+1}$ we generate $e_n - e_{i_0}$ for some $1 \leq i_0 \leq n-1$ as integer combination of $\vec{C}_1, \dots, \vec{C}_k, \vec{C}_{k+1}$.

We are left with the case that $n > r+2$ and

$$\dim(\mathcal{F}_{\mathcal{O}}) \neq \dim(\mathcal{F}_{\mathcal{O}\setminus j}) + 1$$

for all $j \in \{1, \dots, n\}$. As obviously $\dim(\mathcal{F}_{\mathcal{O}}) > \dim(\mathcal{F}_{\mathcal{O}\setminus j})$ Theorem 3.1 implies that $\mathcal{O}$ is not neighborly, but $\mathcal{O} \setminus j$ is neighborly for all $j \in \{1, \dots, n\}$. Therefore, by Lemma 3.5 $n = r+3$. Let $\vec{C}_1, \dots, \vec{C}_{n-2}$ be a basis of $\mathcal{O} \setminus n$. We may assume that $\mathcal{O}$ is oriented such that all circuits in $\mathcal{O} \setminus n$ are balanced. As $n = r+3$, the dual of $\mathcal{O}$ is a rank 3 matroid and, thus has a representation as an oriented pseudo-line configuration. $\mathcal{O} \setminus n$ is represented by some pseudo-line n and, as the vertices on that line correspond to co-circuits of $(\mathcal{O} \setminus n)^*$ they have to be balanced. Accordingly, the orientation of the pseudo-lines has to alternate along n (see Figure 2). Since $\dim(\mathcal{F}) = n$ it must not be possible to orient line n such that all co-circuits in $\mathcal{O}^*$ are balanced. Therefore, there have to be pseudo-lines $i, j, k \in \{1, \dots, n-1\}$ such that the co-circuits $C^*_{i,j}$, $C^*_{i,k}$ formed by the intersections of i and j resp. by i and k share an edge along i and j and k are oriented in the same direction with respect to i. Therefore, if we denote by $C_{n-1} = C^*_{i,j}$ and $C_n = C^*_{i,k}$ the corresponding circuits in $\mathcal{O}$ we have $\vec{C}_{n-1} - \vec{C}_n = \pm(e_j + e_k)$. Using the inductive assumption on $\mathcal{F}_{\mathcal{O}\setminus n}$ and Proposition 3.6 we get all integer vectors with even component sum.

□

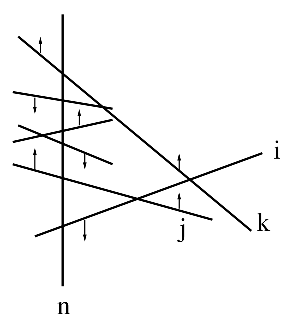


FIGURE 2. Extending an $(r+1)$-element basis of $\mathcal{O}\setminus n$ to a basis of $\mathcal{O}$

This finishes the discussion on the flow lattice of uniform and rank 3 oriented matroids. Summarizing our observations on uniform or rank 3 oriented matroids we find that

– if $\mathcal{O}$ is co-simple and non-regular then $\dim \mathcal{F}_{\mathcal{O}} \in \{n-1, n\}$,
– if $\mathcal{O}$ is non-regular and $\mathcal{F}_{\mathcal{O}}$ is not trivial then $\mathcal{F}_{\mathcal{O}} = v^{\perp} \cap \mathbb{Z}^n$ or $\mathcal{F}_{\mathcal{O}} = \{x \in \mathbb{Z}^n : k | v^T x\}$ for some $v \in \mathbb{Z}^n$,

– if $\mathcal{O}$ is non-uniform and $n > |E(K_r)|$ then $\mathcal{F}_{\mathcal{O}} = \mathbb{Z}^n$,
– $\mathcal{C}(\mathcal{O})$ contains a basis of $\mathcal{F}_{\mathcal{O}}$.

In the next section we give first ideas how far this might extend to general matroids.

6. General Oriented Matroids

The surprising fact that among the considered classes no co-simple oriented matroid has a flow lattice of co-dimension 2 led us to the question what co-dimensions are possible in general. We rule out the trivial cases first. Almost any gap between $\dim \mathcal{F}_{\mathcal{O}}$ and n is possible by sticking together regular and non-regular oriented matroids via direct sum and 2-sum. The famous matroid decomposition theorem into sums and 2-sums of 3-connected minors almost completely generalizes to oriented matroids and the dimension of the lattice is closely related to the dimension of the components. This gives motivation for considering only 3-connected oriented matroids. We have looked at numerous examples and find that our observations from uniform and rank 3 instances generalize at least for small general oriented matroids except for a small family of counterexamples.

6.1. **Decomposition of Oriented Matroids.** Let $\mathcal{O} = (E, \mathcal{C})$. For an edge $f \in E$ we define $\mathcal{C}^{f^0} := \{C \in \mathcal{C} : f \notin C\}$ and $\mathcal{C}^{f^+} := \{C \in \mathcal{C} : f \in C^+\}$. Let $\mathcal{O}_1 = (E_1, \mathcal{C}_1)$ and $\mathcal{O}_2 = (E_2, \mathcal{C}_2)$ be two oriented matroids. If $E_1 \cap E_2 = \emptyset$ then the direct sum $\mathcal{O}_1 \oplus \mathcal{O}_2 = (E_1 \cup E_2, \mathcal{C}_\oplus)$ is defined by the circuits $\mathcal{C}_\oplus := \mathcal{C}_1 \cup \mathcal{C}_2$. Now let $\{f\} = E_1 \cap E_2$ and neither in $\mathcal{O}_1$ nor in $\mathcal{O}_2$ $\{f\}$ is a separator. Then we define the 2-sum $\mathcal{O}_1 \oplus_2 \mathcal{O}_2$ with edge set $(E_1 \cup E_2) \setminus f$ in terms of its circuits

$$\mathcal{C}_{\oplus_2} = \mathcal{C}_1^{f^0} \cup \mathcal{C}_2^{f^0} \cup \{(C_1 \cup C_2) \setminus f : C_1(f) = -C_2(f) \neq 0,\, C_i \in \mathcal{C}_i\}.$$

Proposition 6.1. *$\mathcal{C}_{\oplus_2}$ satisfies the circuit axioms.*

Proof. We set $\tilde{\mathcal{C}} := \{(C_1 \cup C_2) \setminus f : C_1(f) = -C_2(f) \neq 0,\, C_i \in \mathcal{C}_i\}$. Obviously, $\mathcal{C}_{\oplus_2}$ is antisymmetric and its supports form a clutter. It suffices to verify oriented circuit elimination. As this is rather straightforward we consider only the case where $C_1, C_2 \in \tilde{C}$. Thus, let C_{ij}, $i, j \in \{1, 2\}$ be circuits such that $C_i = C_{i1} \cup C_{i2} \setminus f, C_{i1}(f) = -C_{i2}(f)$ where C_{ij} is a circuit in $\mathcal{O}_j$ and w.l.o.g. $f \neq e \in C_{11} \cap C_{21}, C_{11}(e) = -C_{21}(e)$. If $C_{11}(f) = -C_{21}(f)$ then also $C_{12}(f) = -C_{22}(f)$ and eliminating f between C_{12} and C_{22} we find $C_3 \in \mathcal{C}_2^{f^0}$ as required. Otherwise, $C_{11}(f) = C_{21}(f)$ and using circuit elimination fixing f in $\mathcal{O}_1$ we find a circuit $f \in C_{31}$ where $C_{31}^+ \subseteq (C_{11}^+ \cup C_{21}^+) \setminus e$ and $C_{31}^- \subseteq (C_{11}^- \cup C_{21}^-) \setminus e$. Now $C_3 = (C_{31} \cup C_{12}) \setminus f$ settles the case. □

With this construction the decomposition of matroids with low connectivity generalizes to oriented matroids. Surprisingly, this does not seem to have been considered in the literature yet. At least a parallel connection of oriented matroids has been formulated in terms of covectors by Dong [6] very recently.

Let $\mathcal{O}$ be an oriented matroid on a finite set E. Assume the underlying matroid $\mathcal{M}$ has a 2 separation $E = E_1 \cup E_2$. We will use the following Lemma ([17, 8.3.2], [19]).

Lemma 6.2. *Let C and D be circuits of $\mathcal{M}$ that meet both E_1 and E_2. Then $C \cap E_1$ is not a proper subset of $D \cap E_1$.*

Also we need a generalization of Lemma 8.3.3 of [17] (see also [19]). We assume C and D are signed circuits of $\mathcal{O}$ such that $C_i := C \cap E_i \neq \emptyset \neq D \cap E_i =: D_i$ for $i = 1, 2$.

Lemma 6.3. *Either $\vec{C}_1 \circ \vec{D}_2$ or $\vec{C}_1 \circ -\vec{D}_2$ is a signed circuit of $\mathcal{O}$.*

Proof. By Lemma 8.3.3 of [17] $C_1 \cup D_2$ is a circuit in $\mathcal{M}$. Let F denote one of its two orientations in $\mathcal{O}$. Assuming that $S(F, \vec{C}_1) \notin \{\emptyset, C_1\}$ using strong circuit elimination between F and C, eliminating one element from C_1 and fixing another one, we derive a contradiction to Lemma 6.2 ($S(F, \vec{C}_1)$ denotes the separation set of F and $\vec{C}_1$). By symmetry the same holds for D_2 and the claim follows. □

Furthermore, we need the following fact about *co-lines* of oriented matroids, (see [7] for the corresponding result in general matroids).

Proposition 6.4. *Let C be a positive signed circuit of $\mathcal{O}$ and e an element in the matroid closure of C. Then there is a partition $C \cup \{e\} = \gamma_1 \dot\cup \ldots \dot\cup \gamma_k$ where $k \geq 3$ and $\{e\}$ is either γ_1 or γ_k and the signed circuits in $C \cup \{e\}$ are exactly those of the form $(\gamma_1 \cup \ldots \cup \gamma_{i-1}, \gamma_{i+1} \cup \ldots \cup \gamma_k)$ for $i = 1, \ldots, k$ and their negatives.*

Proof. Consider the dual of the restriction of $\mathcal{O}$ to $C \cup \{e\}$. This is a line, consider its affine representation. As C is a co-circuit in this matroid e is either the leftmost or the rightmost point in this representation. As e is in the closure of C in $\mathcal{O}$, clearly $C \cup \{e\}$ does not consist of two disjoint circuits and the claim follows. □

Now we can prove the main lemma for our decomposition result.

Lemma 6.5. *If $\vec{D}_1 \circ \vec{C}_2$ is a circuit then also $\vec{C}_1 \circ \vec{D}_2$ is a circuit.*

Proof. We may assume that $E = C \cup D$ and that $\mathcal{O}$ is oriented such that C is positive. We proceed by induction on $|E|$ and consider three cases.
Case 1: $D_1 = \{e\}$:

Eliminating some element from C_2 between $\vec{C}$ and $-(\vec{D}_1 \circ \vec{C}_2)$ using Lemma 6.2 we find the signed circuit $\vec{C}_1 \circ -\vec{D}_1$, and eliminating e with D we find the signed circuit $\vec{C}_1 \circ \vec{D}_2$ as required.

Case 2: $|D_1| \geq 2$ and there exists an $e \in D_1$ such that $C_1 \cup \{e\}$ is independent in $\mathcal{M}$:

We may assume that $C_1 \neq \{e\}$ and now consider $\mathcal{O}/e$. Here we have the signed circuits $\vec{D} \setminus \{e\}, \vec{C} \setminus \{e\}$ and $\vec{D}_1 \setminus \{e\} \circ \vec{C}_2$. By inductive assumption also $\vec{C}_1 \setminus \{e\} \circ \vec{D}_2$ is an oriented circuit of $\mathcal{O}/e$. Using Lemma 6.3 the claim follows.

Case 3: $|D_1| \geq 2$ and $C_1 \cup \{e\}$ is dependent for all $e \in D_1$:

In particular, this implies $C_1 \cap D_1 = \emptyset$. Fix $e \in D_1$ and let $C \cup \{e\} = \gamma_1 \dot\cup \ldots \dot\cup \gamma_k$ denote the partition as in Proposition 6.4. Eventually re-orienting e we may assume $\gamma_1 = \{e\}$. By Lemma 6.2 there exists an $2 \leq i_0 \leq k$ such that $C_2 \subseteq \gamma_{i_0}$. We consider two subcases:

Case 3.1: $i_0 = 2$:

Let F denote the positive circuit on $\gamma_1 \cup \ldots \cup \gamma_{k-1}$ and $F_1 = F \cap E_1$. Then we have the signed circuits $\vec{F} = \vec{F}_1 \circ \vec{C}_2, \vec{D}$ and $\vec{D}_1 \circ \vec{C}_2$. Using inductive assumption we find the signed circuit $\vec{F}_1 \circ \vec{D}_2$. Eliminating e between F and $(\gamma_3 \cup \ldots, \cup \gamma_k, e)$ we find an oriented circuit that conforms to $\vec{C}_1 \circ \vec{D}_2$ and the claim follows.

Case 3.2: $i_0 > 2$:

Hence $\gamma_2 \cap C_2 = \emptyset$. Let G denote the signed circuit $(\gamma_3 \cup \ldots \cup \gamma_k, e)$ and $G_1 = G \cap E_1$. Again we find signed circuits $\vec{G} = \vec{G}_1 \circ \vec{C}_2, \vec{D}$ and $\vec{D}_1 \circ \vec{C}_2$ implying that $\vec{G}_1 \circ \vec{D}_2$ is a signed circuit. Eliminating e between G and $(\gamma_1 \cup \ldots \cup \gamma_{i_0-1}, \gamma_{i_0+1} \cup \ldots \cup \gamma_k)$ fixing some element from γ_{i_0} we find a circuit in $C_1 \cup D_2$ that conforms to $\vec{D}_2$ and agrees in sign with $\vec{C}_1$ on at least one element. Thus, the claim follows from Lemma 6.3.

$\square$

Now that the main ingredients have been established the argument for a decomposition into 3-connected minors is straightforward.

Theorem 6.6. *Every oriented matroid $\mathcal{O}$ can be decomposed into direct sums and 2-sums of 3-connected oriented matroids.*

Proof. If $\mathcal{O}$ is 3-connected we are done. In case $\mathcal{O}$ is not connected the decomposition of $\mathcal{O}$ into direct sums of 2-connected matroids is also obvious. Now assume $\mathcal{O}$ to be 2-connected and without loss of generality $\underline{\mathcal{O}} = \mathcal{M}_1 \oplus_2 \mathcal{M}_2$ for 2-connected matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on E_1 and E_2 satisfying $E_1 \cap E_2 = \{f\}$. Note, that if f is a loop in one of $\mathcal{M}_1$ or $\mathcal{M}_2$ then $\mathcal{O}$ was not 2-connected.

Choose circuits $(C_i \cup f)$ in $\mathcal{M}_i$ $(i = 1, 2)$ which contain f and another element $g_i \in C_i$. Set $\mathcal{O}_1 := \mathcal{O} \setminus (E_2 - (C_2 \cup f))/(C_2 - \{g_2\})$ and identify g_2 with f (and do this symmetrically for $\mathcal{O}_2$). Reorient g_2 in $\mathcal{O}_2$ so that $\vec{C}_1(g_1) = -\vec{C}_2(g_2)$. Then by [17, Proposition 7.1.19] $\underline{\mathcal{O}_1 \oplus_2 \mathcal{O}_2} = \mathcal{M}_1 \oplus_2 \mathcal{M}_2$. Now, $\vec{C}_1 \circ \vec{C}_2$ is a circuit in $\mathcal{O}_1 \oplus_2 \mathcal{O}_2$ and in $\mathcal{O}$. If $\vec{D} = \vec{D}_1 \circ \vec{D}_2$ and (w.l.o.g.) $\vec{C}_2 \circ \vec{D}_1$ are other circuits in $\mathcal{O}$ then by Lemma 6.5 so must be $\vec{C}_2 \circ \vec{D}_1$. Hence, $\mathcal{O} = \mathcal{O}_1 \oplus_2 \mathcal{O}_2$. $\square$

Remark 6.7.

– *In matroid theory, every matroid can be decomposed into sums and 2-sums of proper minors which is not the case for oriented matroids* *and even not for digraphs which can be seen at the example of the 2-sum of two triangles.*

– *Instead of glueing together circuits with alternating sign in f we could as well compose our sum from circuits with identical signs in f.*

We now analyze the flow lattice dimension of 1- and 2-sums. While the dimension of $\mathcal{F}_{\mathcal{O}_1 \oplus \mathcal{O}_2}$ does not depend on the structure of $\mathcal{O}_1$ and $\mathcal{O}_2$ (*i.e.* $\dim \mathcal{F}_{\mathcal{O}_1 \oplus \mathcal{O}_2} = \dim \mathcal{F}_{\mathcal{O}_1} + \dim \mathcal{F}_{\mathcal{O}_2}$) there are two possibilities for $\dim \mathcal{F}_{\mathcal{O}_1 \oplus_2 \mathcal{O}_2}$:

Lemma 6.8. *If i (*resp. *j) is the column index in $A(\mathcal{C}_1)$ (*resp. *$A(\mathcal{C}_2)$) corresponding to f, then*

$$\dim \mathcal{F}_{\mathcal{O}_1 \oplus_2 \mathcal{O}_2} = \dim \mathcal{F}_{\mathcal{O}_1} + \dim \mathcal{F}_{\mathcal{O}_2} - \begin{cases} 2, & \text{for } e_i \in \operatorname{lin}(\mathcal{C}_1) \text{ and } e_j \in \operatorname{lin}(\mathcal{C}_2), \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Without loss of generality let f be the element with largest index in E_1 and smallest index in E_2. Let furthermore the matrices N_i, M_i and M_1^j be defined as follows: N_i is obtained by deleting the zero column in $A(\mathcal{C}_i^{f^0})$, M_1 is obtained from $A(\mathcal{C}_1^{f^+})$ by deleting the last column, M_2 is obtained from $A(\mathcal{C}_2^{f^-})$ by deleting the first column and M_1^j contains only the j^{th} row of M_1 but $|\mathcal{C}_2^{f^+}|$ times. Then the circuit matrices of $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_{\oplus_2}$ can be written as

$$A(\mathcal{C}_1) = \left(\begin{array}{c|c} N_1 & 0 \\ M_1 & 1 \end{array} \right) \qquad A(\mathcal{C}_{\oplus_2}) = \left(\begin{array}{c|c} N_1 & \mathbf{0} \\ \mathbf{0} & N_2 \\ M_1^1 & M_2 \\ M_1^2 & M_2 \\ \vdots & \vdots \\ M_1^{|\mathcal{C}_1^{f^+}|} & M_2 \end{array} \right)$$

$$A(\mathcal{C}_2) = \left(\begin{array}{c|c} 0 & N_2 \\ -1 & M_2 \end{array} \right)$$

In order to determine the rank of $A(\mathcal{C}_{\oplus_2})$ we add a zero column (for the missing element f) in the middle. Then we add a row, which corresponds to a signed circuit $C \in \mathcal{C}_1$ that contains f, say C corresponds to the circuit in the first row of M_1, and get a matrix B satisfying $\operatorname{rank} B = \operatorname{rank} A(\mathcal{C}_{\oplus_2}) + 1$ (see Figure 3).

By unimodular row operations we get the matrix $\overline{B}$. By the dimension formula of linear algebra we have $\operatorname{rank} \overline{B} = \operatorname{rank} A(\mathcal{C}_1) + \operatorname{rank} A(\mathcal{C}_2) - 1$ if and only if $\operatorname{lin}(\mathcal{C}_1) \cap \operatorname{lin}(\mathcal{C}_2) \neq \emptyset$. This is the case if and only if the unit vector corresponding to f is contained in both $\operatorname{lin}(\mathcal{C}_1)$ and $\operatorname{lin}(\mathcal{C}_2)$. Otherwise $\operatorname{rank} \overline{B} = \operatorname{rank} A(\mathcal{C}_1) + \operatorname{rank} A(\mathcal{C}_2)$. As the rank of $\overline{B}$ is one more than the rank of $A(\mathcal{C}_{\oplus_2})$, the claim follows. □

6.2. **3-connected Oriented Matroids.** By the above we will now restrict our considerations to oriented matroids that must be simple, co-simple, non-regular, non-uniform, and 3-connected. In the remainder we report on computational results on the entire catalog of small oriented matroids (see [8])

$$B = \left(\begin{array}{c|c|c} N_1 & 0 & \mathbf{0} \\ \mathbf{0} & 0 & N_2 \\ M_1^1 & 0 & M_2 \\ \vdots & \vdots & \vdots \\ & & \\ M_1^{|\mathcal{C}_1^{f^+}|} & 0 & M_2 \\ \hline \vec{C} & 1 & 0 \end{array}\right) \rightsquigarrow \overline{B} = \left(\begin{array}{c|c|c} N_1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & 0 & N_2 \\ \hline \mathbf{0} & -1 & M_2 \\ \hline M_1 & 1 & \mathbf{0} \\ \hline \multicolumn{3}{c}{\mathbf{0}} \end{array}\right) \rightsquigarrow \left(\begin{array}{c|c} A(\mathcal{C}_1) & \mathbf{0} \\ \hline \mathbf{0} & A(\mathcal{C}_2) \\ \hline \multicolumn{2}{c}{\mathbf{0}} \end{array}\right)$$

FIGURE 3. Matrices used in the proof of Lemma 6.8

and orientations of interesting matroids selected by [17]. On that data we could verify that

- $\dim \mathcal{F}_\mathcal{O} \in \{n-1, n\}$ for small 3-connected oriented matroids (up to a single counterexample having dimension $n-2$).
- Non-trivial flow lattices seem to vanish for a growing number of elements.
- In any case $\mathcal{F}_\mathcal{O}$ could be characterized very similarly to the uniform case by orthogonality conditions and integral modular equations.

We determined the flow lattice for all known isomorphism classes of non-uniform, non-regular, 3-connected oriented matroids (taken from [8]).

Rank 4, 7 elements: The reorientation classes of a special single element extension of the prism have a flow lattice that is either $(1,\dots,1,0)^\perp \cap \mathbb{Z}^n$ or $(1,\dots,1,2)^\perp \cap \mathbb{Z}^n$ or $\{x \in \mathbb{Z}^n : 2|(1,\dots,1,0)^T x\}$. Up to reorientation, the oriented dual non-Fano matroid $(F_7^-)^*$ yields $\mathcal{F}_\mathcal{O} = (1,0,0,1,0,1,1)^\perp \cap \mathbb{Z}^n$. The remaining flow lattices are all trivial.

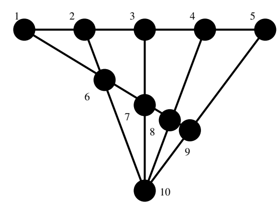
Rank 4, 8 elements: All flow lattices are trivial except for an amalgam of two prisms (the "linear brother" of the Vamós cube) with $\mathcal{F}_\mathcal{O} \cong \{x \in \mathbb{Z}^n : 3|\mathbb{1}^T x\}$. Note, that all flow lattices have full dimension.

Rank 5, 8 elements: We found 81 (1.34%) non-trivial flow lattices of the form $v^\perp \cap \mathbb{Z}^n$ or $\{x \in \mathbb{Z}^n : 2|v^T x\}$ for some $v \in \{0,1,2\}^n$ (up to reorientation) all having dimension $n-1$ or n.

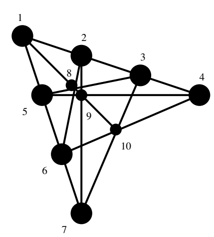
Rank 6, 9 elements: We found 492 (0.1%) non-trivial flow lattices of the form $v^\perp \cap \mathbb{Z}^n$ or $\{x \in \mathbb{Z}^n : 2|v^T x\}$ for some $v \in \{0,1,2\}^n$ (up to reorientation) all having dimension $n-1$ or n. The only exception is the dual of the Pappus having a flow lattice

$$\mathcal{F}_\mathcal{O} \cong \left\{ x \in \mathbb{Z}^n : \begin{array}{c} 2|(1,1,0,0,1,1,1,0,1)^T x \text{ and} \\ (0,1,1,1,1,0,0,1,1)^T x = 0 \end{array} \right\}.$$

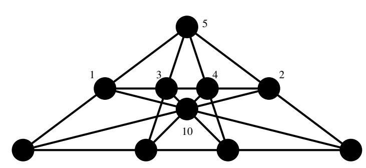
Rank 7, 10 elements: We found 221045 (0.23%) non-trivial flow lattices which again had a simple characterization by a single orthogonality condition or modular equation. Only three matroids had a characterization involving more than one equation (see the configurations of the duals in Figure 4). Note, that the first matroid has flow lattice dimension $n-2$ and can be generalized to a family of examples with this property.



$$\mathcal{F}_{\mathcal{O}^*} \cong \left\{ \begin{array}{c} (1,1,1,1,1,0,0,0,0,0), \\ (1,0,0,0,0,1,1,1,1,0) \end{array} \right\}^{\perp} \cap \mathbb{Z}^n$$



$$\mathcal{F}_{\mathcal{O}^*} \cong \left\{ x \in \mathbb{Z}^n : \begin{array}{r} 2|(0,1,1,0,1,1,0,0,1,1)^T x \text{ and} \\ (0,1,0,1,1,0,1,1,0,1)^T x = 0 \end{array} \right\}$$



$$\mathcal{F}_{\mathcal{O}^*} \cong \left\{ x \in \mathbb{Z}^n : \begin{array}{r} 2|(1,1,0,0,1,0,0,1,1,1)^T x \text{ and} \\ (1,1,1,1,0,1,1,1,1,0)^T x = 0 \end{array} \right\}$$

FIGURE 4. Interesting flow lattices for rank 7 and 10 elements. The figures show a representative of the reorientation class of the dual rank 3 oriented matroid. The flow lattices only apply up to reorientation. The first example and any analogous construction with two lines with an odd number of $2k+1$ ($k \geq 2$) points has flow lattice dimension $n-2$.

$$
\left(\begin{array}{ccccccc|ccccc}
+1 & +1 & +1 & +1 & \dots & +1 & +1 & & & & & \\
+1 & \delta & \delta & \delta & \dots & \delta & \delta & & & & & \\
+1 & +1 & \delta & \delta & \dots & \delta & \delta & & & I_k & & \\
\vdots & \vdots & & & \ddots & \vdots & \vdots & & & & & \\
+1 & +1 & +1 & +1 & \dots & +1 & \delta & & & & & \\
\hline
& & & & & & & \delta\epsilon & -\epsilon & & & \\
& & & & & & & & \epsilon & -\epsilon & & \\
& & & I_k & & & & & & \ddots & \ddots & \\
& & & & & & & & & & \epsilon & -\epsilon \\
& & & & & & & -\epsilon & & & & \epsilon
\end{array}\right)
$$

FIGURE 5. The circuits of the whirl $((\delta, \epsilon) \in \{(+1,+1), (+1,-1), (-1,-1)\})$

[17] listed some interesting matroids but most of them are already contained in [8] or are not orientable, regular, not simple, not co-simple, or not 3-connected. The remaining examples are the whirls $\mathcal{W}^k$. By [16] there are exactly three reorientation classes of the whirl $\mathcal{W}^k$ for $k > 3$ (the circuits are shown in Figure 5). The matrix $A(\mathcal{C}_\mathcal{W})$ is non-singular and has determinant $(-1)^k(1 - \delta\epsilon + \epsilon)^k \in \{1, -1\}$ for $(\delta, \epsilon) \neq (-1, +1)$. Hence, $e_i \in \mathcal{F}_\mathcal{O}$ for $i = 1, \dots, 2k = n$ and the flow lattice is trivial.

6.3. **Open Questions.** What is the dimension of the flow lattice of a general non-uniform non-regular 3-connected oriented matroid? Are there other families of examples having co-dimension 2 or more? Is there a way to characterize $\mathcal{F}_\mathcal{O}$ in terms of a few orthogonality conditions and linear modular equations? Do more complex flow lattices occur for higher rank and cardinality? What is the behavior of $\Phi_\mathcal{L}$ under 2-sums?

Further investigations could be done on the case of co-rank 3. This would require to consider the co-circuits of pseudo-line arrangements. Is it possible to construct a counterexample to the question of [14] "Is $\Phi_\mathcal{L}$ a matroid invariant?" by exploiting this geometric instrument? As we could see above this at least holds for uniform, rank 3, and small general oriented matroids.

Another interesting question involves the gap in the flow lattice dimension between regular and non-regular oriented matroids: Does the dimension always increase rapidly when extending a regular oriented matroid non-regularly? To be more offensive, is the flow lattice of a simple and co-simple extension of a maximal regular oriented matroid always trivial (we considered $\mathcal{M}(K_4)$ and checked the extensions of $\mathcal{M}(K_5)$ with a software of O. Klein [15])? If $\mathcal{O}$ is not maximal regular we already know from the single element extensions of the prism that the dimension might grow only by r instead of $r + 1$.

For regular, uniform, and rank 3 oriented matroids one can construct a basis of $\mathcal{F}_\mathcal{O}$ containing only signed circuits. Is this possible in general and is there a way to compute the basis without the knowledge of the entire set $\mathcal{C}$? (The number of circuits rapidly increases with growing cardinality.)

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