A CANONICAL PARTITION THEOREM FOR TREES

VOJKAN VUKSANOVIC

ABSTRACT. We show that for every positive integer d, every downwards closed subtree T of $d^{<\mathbb{N}}$ without terminal nodes and every finite weakly embedded subtree A of T there is a finite list of equivalence relations on $Em^A(T)$ with the property that for every other equivalence relation $\mathbb{E}$ on $Em^A(T)$ there is a strongly embedded subtree $S \subseteq T$ of height ω, such that $\mathbb{E} \restriction Em^A(S)$ is equal to one of the equivalence relations from the list.

1. INTRODUCTION

Given a set X with some topology and/or structure one frequently needs to understand which kind of equivalence relations one can have on X as well as their behaviour on substructures that are similar to X in some way. In the literature one can find several approaches in classifying equivalence relations or the corresponding quotient structures. The approach we follow has been started by Erdős and Rado [1] for the case when the set X is equal to the symmetric cube

$$[\mathbb{N}]^n = \{F \subseteq \mathbb{N} : |F| = n\}$$

of the set $\mathbb{N}$ of positive integers. Note the following family $\mathbb{E}^n_\Delta$ ($\Delta \subseteq \{1, ..., n\}$) of "canonical" equivalence relations one can define on $[\mathbb{N}]^n$:

$$\{x_1, ..., x_n\} \ \mathbb{E}^n_\Delta \ \{y_1, ..., y_n\} \ \text{ if and only if } \ x_i = y_i \text{ for all } i \in \Delta,$$

where we assume that the n-element sets are listed in the increasing order. What Erdös and Rado [1] prove is that $\mathbb{E}^n_\Delta$ ($\Delta \subseteq \{1, ..., n\}$) forms a basis for the family of all equivalence relations on the symmetric cube $[\mathbb{N}]^n$ in a very precise sense: for every equivalence relation $\mathbb{E}$ on $[\mathbb{N}]^n$ there exists an infinite $M \subseteq \mathbb{N}$ and $\Delta \subseteq \{1, ..., n\}$ such that

$$\mathbb{E} \restriction [M]^n = \mathbb{E}^n_\Delta \restriction [M]^n.$$

They have concentrated on the classification of the equivalence relations on $[\mathbb{N}]^n$ rather than $\mathbb{N}^n$ because they considered their result as an extension of the famous Ramsey's theorem [4] which in this terminology simply asserts that the full and empty equivalence relations on $[\mathbb{N}]^n$ form a 2-element basis

for the class of all equivalence relations on $[\mathbb{N}]^n$ with finitely many classes. However, it is not hard to see that one can actually obtain basis for the class of all equivalence relations on $\mathbb{N}^n$ as well. In fact the existence of the finite basis for equivalence relations on $\mathbb{N}^n$ can be easily deduced from the Erdős-Rado result about $\mathbb{E}_\Delta^k$ ($\Delta \subseteq \{1, ..., k\}, 1 \leq k \leq n$).

In this paper we consider downwards closed subtrees T of $d^{<\mathbb{N}}$ of height ω without terminal nodes and prove a canonical partition result for the set of all finite weakly embedded subtrees of T which have the same embedding type (see Definition 2.3) as A where A is a given finite subtree of T closed under the meet operation. This is an extension of the result of Milliken (see [3]) about weakly embedded subtrees of rooted, finitely branching trees, without terminal nodes and of height ω. We are using the same method as in [6] (see also [7] and [5]). The author has learned that there is an unpublished paper of Milliken ([2]), circulated in the 1980's, about canonical partitions of finite strongly embedded subtrees of regular trees. Although one can prove the result about finite weakly embedded subtrees from it, in this paper we give a direct proof for any finite weakly embedded subtree of a given regular tree.

2. Preliminaries

A tree is any partially ordered set $(T, \leq)$ such that for any $t \in T$ the set $\{s \in T : s \leq t\}$ is well ordered by the induced order. Given a tree T we say that S is a subtree of T if $S \subseteq T$. From now on we will suppose that every tree has a root, *i.e.* we will suppose that every tree has a unique minimal element. By $Succ(t, T)$ we will denote the set of all $s \in T$ such that $t \leq s$. By $Pred(t, T)$ we will denote the set of all $s \in T$ such that $s \leq t$ and by $IS(t, T)$ we will denote the set of immediate successors of t in T. We will also suppose that for a given tree T, for every $t \in T$, both $Pred(t, T)$ and $IS(t, T)$ are finite. We say that a tree T is *regular* if there is a positive integer d such that for all $t \in T$, $|IS(t, T)| = d$, and we call d *degree of* T. By $T(n)$ we will denote the n-th level of T i.e. the set of all $t \in T$ such that $|\{s \in T : s \leq t, s \neq t\}| = n$. For a tree T, $height(T)$ will denote $\sup\{n + 1 : T(n) \neq \emptyset\}$. Given a tree T and nodes s and t in T by $s \wedge t$ we will denote the maximal node in T which is below both s and t. If $A \subseteq T$, by $\wedge(A)$ we will denote the $\wedge$ - closure of A, the smallest subset A' of T containing A such that for all $s, t \in A$ the node $s \wedge t$ belongs to A'.

Definition 2.1. *Suppose that S is a subtree of T. We say that S is strongly embedded in T (see Figure 1 for the case $T = 3^{<\mathbb{N}}$) if:*

   (1) *If $s \in S(n)$ for some $n < height(S) - 1$ and $t \in IS(s, T)$, then $Succ(t, T) \cap IS(s, S)$ is a singleton,*
   (2) *If $height(T) = \omega$ and $height(S) \leq \omega$ there is a strictly increasing function $f : height(S) \to \omega$ such that $S(n) \subseteq T(f(n))$ for each $n \in height(S)$.*
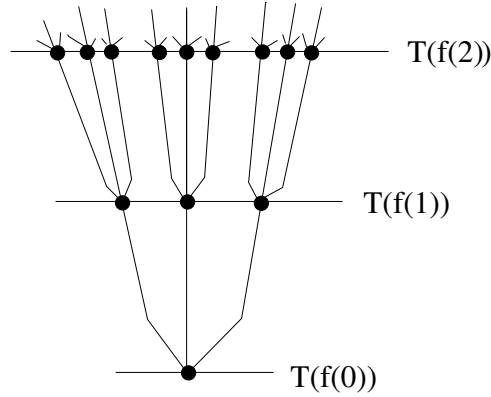
T(f(2))

T(f(1))

T(f(0))

FIGURE 1

If in the previous definition we drop requirement 2), and instead of 1) we have:

    1'. If s is a nonmaximal element of S and $t \in IS(s, T)$, then $Succ(t, T) \cap IS(s, S)$ is either a singleton or empty.

then we say that S is *weakly embedded* in T.

From now on by $WEm^{<\omega}(T)$ we denote the set of all finite weakly embedded subtrees of T.

Note that a strongly embedded subtree S' of a strongly embedded subtree S of T is strongly embedded in T, and a weakly embedded subtree A of a strongly embedded subtree S of T is weakly embedded in T.

Lemma 2.2. *Let d be a positive integer and let T be a strongly embedded subtree of $d^{<\mathbb{N}}$. Then if $height(T) = n$ $(height(T) = \omega)$ there is a unique bijection $h : d^{\leq n-1} \to T$ (resp. $h : d^{<\mathbb{N}} \to T$) such that:*

    (1) *for every $s, t \in d^{<\mathbb{N}}$, $s \subseteq t \Leftrightarrow h(s) \subseteq h(t)$,*

    (2) *for every $s, t \in d^{<\mathbb{N}}$, $|s| < |t| \Leftrightarrow |h(s)| < |h(t)|$ and*

    (3) *for every $s, t \in d^{<\mathbb{N}}$, $s <_{lex} t \Leftrightarrow h(s) <_{lex} h(t)$, where $<_{lex}$ denotes the lexicographic ordering of $d^{<\mathbb{N}}$.*

Proof. We will prove the lemma by induction on $i < height(T)$. In the case $height(T) = 1$ there is nothing to prove because T is a single node and $d^0 = \{\emptyset\}$. So, suppose that the lemma is true for all strongly embedded subtrees of height $\leq n$ and let T be a strongly embedded subtree of $d^{<\mathbb{N}}$ of height $n + 1$. Let $T' = \cup_{i=0}^{n-1} T(i)$. Then T' is a strongly embedded subtree of $d^{<\mathbb{N}}$ of height n. By the inductive hypothesis there is a unique bijection $h' : d^{\leq n-1} \to T'$ fulfilling the requirements 1-3 of the lemma. Define $h : d^{\leq n} \to T$ as follows: put $h \upharpoonright T' = h'$ and for every $t \in T(n)$ if t is an i-extension of some $t' \in T(n-1)$ for $0 \leq i \leq d-1$ then put $h(t) = h'(t')\hat{\ }i$. It is clear that h is a bijection between T and $d^{\leq n}$ which satisfies requirements 1-3 of the lemma. The uniqueness follows from the uniqueness of the restriction h' and the fact that any other bijection g which

satisfies the requirements 1-3 of the lemma must also satisfy $g(t) = h'(t')^\frown i$ if $t \in T(n)$ is an i-extension of some $t' \in T(n-1)$.

If T is a strongly embedded subtree of $d^{<\mathbb{N}}$ of height ω then for each positive integer n, $T_n = \cup_{i=0}^{n-1} T(i)$ is a strongly embedded subtree of $d^{<\mathbb{N}}$ of height n. So, for each positive integer n there is a bijection $h_n : d^{\leq n-1} \to T_n$ satisfying requirements 1-3 of the lemma. By the first part of the proof for every $1 \leq m < n$ we have that $h_n \upharpoonright T_m = h_m$. Hence, $h = \cup_{n=1}^\infty h_n$ is a bijection between T and $d^{<\mathbb{N}}$ which satisfies the requirements 1-3 of the lemma. The uniqueness of h follows from the fact that $h \upharpoonright T_n$ is unique for every positive integer n. This finishes the proof of the lemma.

$\square$

We call h the *standard tree isomorphism of T*.

From now on by T we denote a downwards closed subtree of $d^{<\mathbb{N}}$ of height ω and without terminal nodes.

Definition 2.3. *Given a strongly embedded subtree S of T and trees A and B finite, weakly embedded in S, we say that A and B have the same embedding type and we write $A \sim_{Em} B$ provided the following hold:*

(1) *There is a bijection $f : A \to B$ satisfying $a \subseteq a'$ if and only if $f(a) \subseteq f(a')$,*

(2) *If $a \in A \cap T(n), a' \in A \cap T(n'), f(a) \in B \cap T(m)$, and $f(a') \in B \cap T(m')$ then $n < n'$ if and only if $m < m'$ and*

(3) *Let $a, a' \in A$ be such that $|a| < |a'|$. Then we require that $a'(|a|) = i$ if and only if $f(a')(|f(a)|) = i$ where $0 \leq i \leq d - 1$.*

We write $Em^A(T)$ for the collection of all weakly embedded subtrees B of T with $A \sim_{Em} B$.

Note that the above definition is not the same as the original definition of Milliken given in [3]. Milliken required the immediate successors of each node in the finitely branching tree T be linearly ordered by some $\prec$. In other words, for each $t \in T$, $IS(t, T)$ can be enumerated as

$$IS(t, T) = \{is(t, T)(j) : j \in |IS(t, T)|\}$$

so that

$$is(t, T)(i) \prec is(t, T)(j) \Leftrightarrow i \in j \in |IS(t, T)|.$$

The third condition of Milliken's definition asserts that given $a \in A$ on level n and a' on a higher level, with $f(a) \in B$ on level m and $f(a') \in B$ on a higher level, if b is the element of level $n + 1$ which is below a' in the tree and b' is the element of level $m + 1$ which is below $f(a')$ in the tree order, then the equivalence $b = is(a, T)(i)$ if and only if $b' = is(f(a), T)(i)$. In the case of downwards closed subtrees of $d^{<\mathbb{N}}$ of height ω and without terminal nodes we naturally take the lexicographic ordering to be the linear order $\prec$. Then it is easy to see that Definition 2.3 is equivalent to the original one when working with the strongly embedded subtrees of $d^{<\mathbb{N}}$.

Lemma 2.4. *There are $1 + d + (1 + 2d)\binom{d}{2}$ different embedding types which appear as the meet closure of a pair in $d^{<\mathbb{N}}$.*

Proof. Figure 2 illustrates the case $d = 3$. There is one embedding type for degenerate pairs consisting of a single point. There are d embedding types for pairs $\{s, t\}$ with $s \subsetneq t$. There are $\binom{d}{2}$ embedding types for incomparable pairs $\{s, t\}$ with $|s| = |t|$. Finally there are $2d\binom{d}{2}$ embedding types for incomparable pairs $\{s, t\}$ with $s <_{lex} t$ and either $|s| < |t|$ or $|s| > |t|$. □
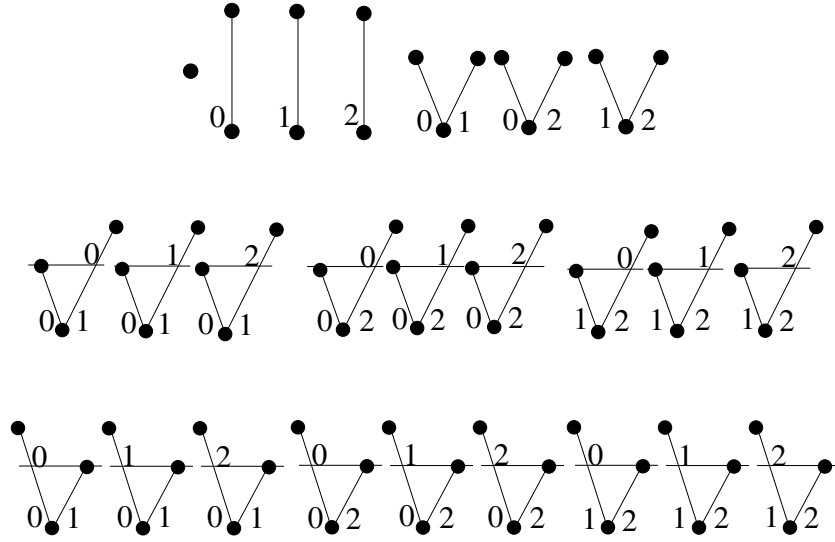

FIGURE 2

Lemma 2.5. *Let S be a strongly embedded subtree of T and let $A, B \in WEm^{<\omega}(S)$ be such that $A \sim_{Em} B$. Then there is only one bijection f between A and B witnessing $A \sim_{Em} B$.*

Proof. Use induction on the number of elements of A. □

Lemma 2.6. *Let T be a strongly embedded subtree of $d^{<\mathbb{N}}$ and let Φ_T be the standard tree isomorphism of T. For every weakly embedded subtree S of T, we have $\Phi_T^{-1}{}''S \sim_{Em} S$. Moreover, among the subtrees of $d^{\leq n-1}$, there are realizations of all embedding types of weakly embedded subtrees of $d^{<\mathbb{N}}$ which have at most n elements.*

Proof. Use the definitions of embedding type and tree isomorphism to prove that $\Phi_T^{-1}{}''S \sim_{Em} S$.

Given a weakly embedded subtree S of T with $m \leq n$ nodes, build a strongly embedded subtree U of T such that $S \subseteq U$ and $\{m : S \cap T(m) \neq \emptyset\} = \{m : U \cap T(m) \neq \emptyset\}$, and observe that $height(U) \leq m \leq n$. Let Φ_U be the standard tree isomorphism of U. Then $\Phi_U^{-1}{}''S$ is a subtree of $d^{\leq m-1} \subseteq d^{\leq n-1}$ of the same embedding type as S. □

The following theorem, which is due to Milliken [3], is crucial for our proof.

Theorem 2.7. *Let S be a strongly embedded subtree of T of height ω, $A \in WEm^{<\omega}(S)$ and $Em^A(S) = \bigcup_{i \in r} C_i$ for some positive integer r. Then there is a strongly embedded subtree U of S of height ω and $k \in r$ such that $Em^A(U) \subseteq C_k$.*

3. Transitive Sets

From now on let F be a fixed weakly embedded subtree of a given tree T. we shall proceed now as we did in [6], [7] and [5].

Definition 3.1. *Let $A, B, C, D \in Em^F(T)$. Then we write $A : B = C : D$ if there is a bijection $f : \wedge(A \cup B) \to \wedge(C \cup D)$ which preserves the embedding type such that $f''A = C$ and $f''B = D$.*

From now on, whenever we write $A : B = C : D$ the requirements mentioned in the first sentence of Definition 3.1 are to be understood.

Definition 3.2. *Let $A, B, C, D \in Em^F(T)$. Define $A : B \simeq C : D$ if and only if $A : B = C : D$ or $A : B = D : C$.*

Lemma 3.3. *For each finite weakly embedded subtree F of T there is a finite set $\Lambda_F \subseteq Em^F(T) \times Em^F(T)$ such that for every $(A, B) \in Em^F(T) \times Em^F(T)$ there is a pair $(C, D) \in \Lambda_F$ such that*

$$A : B \simeq C : D.$$

Proof. The proof follows immediately from Definition 3.1 and from the fact there are just finitely many embedding types which appear as the $\wedge$-closure of the union of two finite subtrees of T which have the same embedding type as F. Indeed, given $A, B \in Em^F(T)$ we have $n \leq |A \cup B| \leq 2n$, where $|F| = n$. By Lemma 2.6 all embedding types of the weakly embedded subtrees of T which have $\leq 2n$ elements can be realized in $d^{\leq 4n-2}$. □

Let, from now on, $\Lambda_F \subseteq Em^F(T) \times Em^F(T)$ be a fixed minimal set as in the previous lemma.

Definition 3.4. *Let S be a strongly embedded subtree of T of height ω and let $F \in WEm^{<\omega}(S)$. We say that $\mathcal{T} \subseteq \Lambda_F$ is transitive for S if*

  (1) *It contains a pair (A, A) where $A \in Em^F(S)$ and*
  (2) *For every $(A_1, B_1), (A_2, B_2) \in \mathcal{T}$ there is a pair $(A_3, B_3) \in \mathcal{T}$ such that if $C, D, E \in Em^F(S)$ and $C : D \simeq A_1 : B_1$ and $D : E \simeq A_2 : B_2$ then $C : E \simeq A_3 : B_3$.*

Let S be a strongly embedded subtree of T of height ω, let $F \in WEm^{<\omega}(S)$ and let $\mathcal{T} \subseteq \Lambda_F$ be transitive for S. Given $A, B \in Em^F(S)$ set:

$A \; \mathbb{E}_{\mathcal{T}} \; B$ if and only if $A : B \simeq C : D$ for some $(C, D) \in \mathcal{T}$.

Lemma 3.5. $\mathbb{E}_{\mathcal{T}}$ *is an equivalence relation on* $Em^F(S)$. *Also, different transitive sets for* S *define different equivalence relations.*

Proof. 1. Reflexivity and symmetry of $E_{\mathcal{T}}$ follow immediately from the definition of $E_{\mathcal{T}}$.

2. To prove transitivity, suppose $A \ \mathbb{E}_{\mathcal{T}} \ B$ and $B \ \mathbb{E}_{\mathcal{T}} \ C$. If $A = B$ or $B = C$ then $A \ \mathbb{E}_{\mathcal{T}} \ C$, so suppose that $A \neq B \neq C$. Then there are (A_1, B_1) and (A_2, B_2) in $\mathcal{T}$ such that $A_1 : B_1 \simeq A : B$ and $A_2 : B_2 \simeq B : C$. By Definition 3.3 there is a pair $(A_3, B_3) \in \mathcal{T}$ such that $A : C \simeq A_3 : B_3$. This implies $A \ \mathbb{E}_{\mathcal{T}} \ C$. Hence, $\mathbb{E}_{\mathcal{T}}$ is transitive.

To finish the proof of the lemma, suppose that $\mathcal{T}'$, $\mathcal{T}''$ are two different transitive sets for S. Then either there is a pair $(A', B') \in \mathcal{T}' \setminus \mathcal{T}''$ or there is a pair $(A'', B'') \in \mathcal{T}'' \setminus \mathcal{T}'$. The assertion of the theorem now follows immediately from the definition of $\mathbb{E}_{\mathcal{T}'}$ and $\mathbb{E}_{\mathcal{T}''}$. $\qquad\square$

Given a strongly embedded subtree S of T of height ω, $F \in WEm^{<\omega}(S)$ and a transitive set $\mathcal{T} \subseteq \Lambda_F$, from now on we will denote by $\mathbb{E}_{\mathcal{T}}$ the relation on $Em^F(S)$ which corresponds to $\mathcal{T}$. We call these equivalence relations $\mathbb{E}_{\mathcal{T}}$, *canonical relations*. We will prove now that the name canonical is appropriate.

Theorem 3.6. *Let* S *be a strongly embedded subtree of* T *of height* ω *and* $F \in WEm^{<\omega}(S)$. *Then for every equivalence relation* $\mathbb{E}$ *on* $Em^F(S)$ *there is a strongly embedded subtree* U *of* S *of height* ω *and there is* $\mathcal{T} \subseteq \Lambda_F$ *transitive for* U *such that for every* $A, B \in Em^F(U)$, *we have*

$$A \ \mathbb{E} \ B \ \Leftrightarrow \ A \ \mathbb{E}_{\mathcal{T}} \ B.$$

Proof. Without loss of generality we may assume that S has the property that every strongly embedded subtree of S of height ω realizes the F-embedding type. Otherwise, choose a strongly embedded subtree U of S of height ω which does not realize the F-embedding type. Then U trivially satisfies the theorem. Let $\{(A_1, B_1), ..., (A_l, B_l)\} \subseteq \Lambda_F$ be a minimal set such that for every $C, D \in Em^F(S)$ there is a pair (A_i, B_i) from the list such that $A_i : B_i \simeq C : D$. By induction on $1 \leq i \leq l$, we will find a decreasing sequence $\{T_i : 1 \leq i \leq l\}$ of strongly embedded subtrees of S of height ω such that for every $1 \leq i \leq l$ the following holds:

$\star(i)$: for every $1 \leq j \leq i$ we have either $C \ \mathbb{E} \ D$ for every $C, D \in Em^F(T_i)$ such that $C : D \simeq A_j : B_j$ or $C \ \mathbb{E} \hspace{-0.9em}/ \ \ D$ for every $C, D \in Em^F(T_i)$ such that $C : D \simeq A_j : B_j$.

The initial step $i = 1$: Define

$$g_1 : Em^{\wedge(A_1 \cup B_1)}(S) \to \{0, 1\}$$

by $g(H) = 1$ if and only if $C \ \mathbb{E} \ D$ where $H \in Em^{\wedge(A_1 \cup B_1)}(S)$ and C and D are unique n-element subsets of H such that $A_1 : B_1 = C : D$. By Theorem 2.7 there is a strongly embedded subtree T_1 of S of height ω such that g_1 is monochromatic on $Em^{\wedge(A_1 \cup B_1)}(T_1)$. This means that we have

either $C \mathbb{E} D$ for every $C, D \in Em^F(T_1)$ such that $C : D \simeq A_1 : B_1$ or $C \not\mathbb{E}$ D for every $C, D \in Em^F(T_1)$ such that $C : D \simeq A_1 : B_1$. This finishes the proof of the initial step $i = 1$.

The induction step for i with $1 < i < l$: Suppose that we have defined a sequence $\{T_j : 1 \le j \le i\}$ for $1 \le i < l$ which satisfies $\star(i)$. Define

$$g_{i+1} : Em^{\wedge(A_{i+1} \cup B_{i+1})}(T_i) \to \{0, 1\}$$

with $g_{i+1}(H) = 1$ if and only if $C \mathbb{E} D$ where $F \in Em^{\wedge(A_{i+1} \cup B_{i+1})}(T_i)$ and C and D are unique n-element subsets of H such that $A_{i+1} : B_{i+1} = C : D$. By Theorem 2.7 there is a strongly embedded subtree T_{i+1} of T_i of height ω such that g_{i+1} is monochromatic on $Em^{\wedge(A_{i+1} \cup B_{i+1})}(T_{i+1})$. Therefore, either $C \mathbb{E} D$ for every $C, D \in Em^F(T_{i+1})$ such that $C : D \simeq A_{i+1} : B_{i+1}$ or $C \not\mathbb{E}$ D for every $C, D \in Em^F(T_{i+1})$ such that $C : D \simeq A_{i+1} : B_{i+1}$. Since T_{i+1} is a strongly embedded subtree of T_i of height ω, it is strongly embedded in S and also of height ω. By the inductive hypothesis for every $1 \le j \le i$ we have either $C \mathbb{E} D$ for every $C, D \in Em^F(T_{i+1})$ such that $C : D \simeq A_j : B_j$ or $C \not\mathbb{E} D$ for every $C, D \in Em^F(T_{i+1})$ such that $C : D \simeq A_j : B_j$. Hence T_{i+1} satisfies $\star(i + 1)$.

Let $U = T_l$ and $\mathcal{T} = \{(A, B) \in \Lambda_F : \text{there is } C, D \in Em^F(U) \text{ such that } A : B \simeq C : D \text{ and } C \mathbb{E} D\}$.

Let us check that $\mathcal{T}$ is as claimed in the theorem. First, let us prove that $\mathcal{T}$ is transitive for U. It is obvious that $\mathcal{T}$ contains a pair (A, A) such that $A \in Em^F(U)$. Let $(A_1, B_1), (A_2, B_2) \in \mathcal{T}$ be arbitrary and let $C, D, E \in Em^F(U)$ be such that $C : D \simeq A_1 : B_1$ and $D : E \simeq A_2 : B_2$. By the definition of the set $\mathcal{T}$ we must have $C \mathbb{E} D$ and $D \mathbb{E} E$. Hence $C \mathbb{E} E$. Let $(A_3, B_3) \in \Lambda_F$ be such that $A_3 : B_3 \simeq C : E$. Hence, $\mathcal{T}$ is transitive.

Let $A, B \in Em^F(U)$. Suppose that $A \mathbb{E}_{\mathcal{T}} B$. Then, there is a pair $(C, D) \in \mathcal{T}$ such that $A : B \simeq C : D$. Then by the definition of the set $\mathcal{T}$ there are $A', B' \in Em^F(U)$ such that $C : D \simeq A' : B'$ and $A' \mathbb{E} B'$. But, by the construction of the subtree U we must have $A \mathbb{E} B$. This finishes the proof of the theorem. $\qquad\square$

As in the case of rationals (see [7]) and random graphs (see [5]) there is a more "canonical" way to describe transitive sets. Before we state the result we need the following definitions.

Definition 3.7. *Let $F \in WEm^{<\omega}(T)$. By F^* we denote the set of all $t \in T$ such that there is a positive integer n and there are $s, u \in F$ with $t, u \in T(n)$ and $t \subseteq s$.*

Lemma 3.8. *Let $A, B \in WEm^{<\omega}(T)$ be such that $A \simeq B$. Then $A^* \simeq B^*$. Moreover, if $f : A^* \to B^*$ is a bijection witnessing $A^* \simeq B^*$, then $f \upharpoonright A$ is a bijection from A onto B witnessing $A \simeq B$.*

Proof. The proof follows immediately from the definition of embedding type. $\qquad\square$

Given $F \in WEm^{<\omega}(T)$, by (N, L) we denote an arbitrary pair such that $N \subseteq F^*$, $L \subseteq \{0, 1, ..., height(F^*) - 1\}$, $N = \wedge(N)$, and if $n < height(F^*)$ is a maximal integer such that there is a node $t \in N$ with $t \in F^*(n)$ then $n < min(L)$. We call the pair (N, L), a *node-level pair of F*.

Definition 3.9. *Let $F \in WEm^{<\omega}(T)$, let (N, L) be a node-level pair of F and let $A, B \in Em^F(T)$. We write $A \upharpoonright (N, L) = B \upharpoonright (N, L)$ providing the following holds: if $f : F^* \to A^*$ and $g : F^* \to B^*$ are the bijections witnessing $F^* \sim_{Em} A^*$ and $F^* \sim_{Em} B^*$ then:*

> *$f(t) = g(t)$ for every $t \in N$, and for every $n \in L$, if $A^*(n) \subseteq T(n')$ and $B^*(n) \subseteq T(n'')$, then $n' = n''$.*

Remark 3.10. *Note that if $S \subseteq T$ is a finite strongly embedded subtree of T then $S^* = S$. Moreover, if (N, L) is a node-level pair for S and if $U, V \in Em^S(T)$ then $U \upharpoonright (N, L) = V \upharpoonright (N, L)$ means simply that the corresponding nodes of U and V are equal and $U(i)$ and $V(i)$ lie on the same level of T for every $i \in L$.*

From now on let T be a regular tree of degree d. Very often in the proof that follows we will use the procedure of *stretching the subtree*. By this we mean the following: Let n, k be positive integers, let $F = \{f_1, ..., f_n\} \in WEm^{<\omega}(T)$ and let $f_i \in F$ be fixed. One can easily construct a subtree $H = \{h_1, ..., h_n, h_i^1, ..., h_i^k\}$ of T such that $H' \sim_{Em} F$, where $H' = \{h_1, ..., h_n\}$, and $(H' \backslash \{h_i\}) \cup \{h_i^j\} \sim_{Em} F$ for each $1 \leq j \leq k$. We also need the following definition.

Definition 3.11. *Let $\mathcal{T} \subseteq \Lambda_F$ be a transitive set. We say that a node-level pair (N, L) of F is* maximal for $\mathcal{T}$ *if $(N, \emptyset)$ is maximal such that for all $(A, B) \in \mathcal{T}$, $A \upharpoonright (N, \emptyset) = B \upharpoonright (N, \emptyset)$ and having defined such an N, L is maximal among subsets for which (N, L) is a node-level pair of F and for all $(A, B) \in \mathcal{T}$, $A \upharpoonright (N, L) = B \upharpoonright (N, L)$.*

Then we have the following theorem.

Theorem 3.12. *Let $F \in WEm^{<\omega}(T)$, let $\mathcal{T} \subseteq \Lambda_F$ be a transitive set for T and let a node-level pair (N, L) of F be maximal for $\mathcal{T}$. Then*

$$\mathcal{T} = \{(A, B) \in \Lambda_F : A \upharpoonright (N, L) = B \upharpoonright (N, L)\}.$$

Proof. We will prove the theorem by induction on the number of elements of F^*. From now on by n we denote the number of elements of F^*.

The initial step $n = 1$: Let $F = \{\emptyset\}$ and let Λ_F be a minimal set as in Lemma 3.3. By replacing (A, B) by (B, A) if necessary, we may assume without loss of generality that for all $(A, B) = (\{a\}, \{b\}) \in \Lambda_F$ with $a \neq b$ one of the following conditions holds: $a \subseteq b$ or (a and b are incomparable and $a <_{lex} b$). Since $\mathcal{T}$ is transitive, $(\{\emptyset\}, \{\emptyset\})$ is in $\mathcal{T}$. We can now describe the possibilities for $\mathcal{T}$.

Claim 3.12.a. *The set $\mathcal{T}$ is one of the following:*

(1) $\mathcal{T} = \{(\{\emptyset\}, \{\emptyset\})\}$;
(2) $\mathcal{T} = \{(\{p\}, \{q\}) \in \Lambda_F : |p| = |q|\}$; or
(3) $\mathcal{T} = \Lambda_F$.

Proof of Claim. The proof proceeds by a case analysis.
Case 1: $|\mathcal{T}| = 1$:
  Then $\mathcal{T} = \{(\{\emptyset\}, \{\emptyset\})\}$.

Case 2: $|\mathcal{T}| > 1$ and for all $(\{p\}, \{q\}) \in \mathcal{T}$, $|p| = |q|$:
  Let $(A, B) = (\{a\}, \{b\}) \in \mathcal{T} \setminus \{(\{\emptyset\}, \{\emptyset\})\}$ be arbitrary. Then $|a| = |b|$ and by our assumption on Λ_F, $a <_{lex} b$. Let $0 \le i < j \le d-1$ be such that $(a \wedge b)\hat{\ }i \subseteq a$ and $(a \wedge b)\hat{\ }j \subseteq b$. Then the embedding type of the meet closure of $\{a, b\}$ is sketched in Figure 3.
  Suppose $(P, Q) = (\{p\}, \{q\}) \in \Lambda_F$ is such that $|p| = |q|$. Then by our assumption about Λ_F, $p <_{lex} q$. Let $0 \le i' < j' \le d-1$ be such that $(p \wedge q)\hat{\ }i' \subseteq p$ and $(p \wedge q)\hat{\ }j' \subseteq q$.
  Since T is a regular tree, it follows from Lemma 2.6 that S has subtrees of every finite weak embedding type. Thus there are $c, e, f \in T$ such that the meet closure of $\{c, e, f\}$ is a subtree of T with the weak embedding type of the tree sketched in Figure 3 whose leaf nodes are labeled with $C = \{c\}$, $E = \{e\}$ and $F = \{f\}$.
  Since $C : E \simeq A : B$, $E : F \simeq A : B$ and $C : F \simeq P : Q$, by the transitivity of $\mathcal{T}$, it follows that $(P, Q) \in \mathcal{T}$. Since (P, Q) was arbitrary, it follows that $\mathcal{T} = \{(\{p\}, \{q\}) \in \Lambda_F : |p| = |q|\}$.
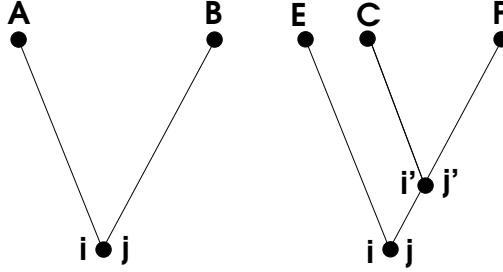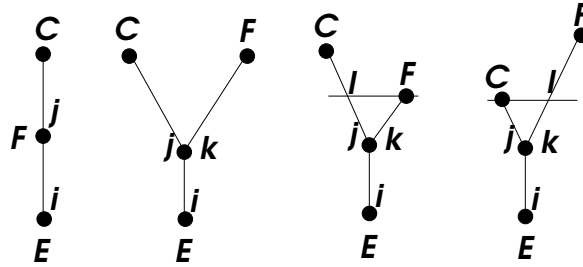

FIGURE 3

Case 3: There is $(A, B) = (\{a\}, \{b\}) \in \mathcal{T}$ with $a \subsetneq b$ or $b \subsetneq a$:
  By our assumption about Λ_F we know that $a \subsetneq b$. Let $0 \le i \le d-1$ be such that $a\hat{\ }i \subseteq b$.
  We shall show that in this case, $\mathcal{T} = \Lambda_F$. Suppose $(P, Q) = (\{p\}, \{q\})$ is an arbitrary element of $\Lambda_F \setminus \{(\{\emptyset\}, \{\emptyset\})\}$. If $p \subsetneq q$, let j be such that $p\hat{\ }j \subseteq q$. If p and q are incomparable, then let $0 \le j < k \le d-1$ be such that $(p \wedge q)\hat{\ }j \subseteq p$ and $(p \wedge q)\hat{\ }k \subseteq q$. If $|p| < |q|$, then let $l = q(|p|)$. The weak embedding type of the meet closure of $\{p, q\}$ is determined by these parameters.

As noted in the previous case, T realizes every finite weak embedding type. First suppose $p \subsetneqq q$. Then there are $c, e, f \in T$ such that the meet closure of $\{c, e, f\}$ is a subtree of T with the weak embedding type of the leftmost tree sketched in Figure 4 whose leaf nodes are labeled with $C = \{c\}$, $E = \{e\}$ and $F = \{f\}$ and $C : F \simeq P : Q$. Since $C : D \simeq A : B$ and $D : E \simeq A : B$, by the transitivity of $\mathcal{T}$, it follows that $(P, Q) \in \mathcal{T}$. Next suppose that $p \not\subseteq q$. Then for one of the other trees sketched in Figure 4, the embedding type of the meet closure of the pair of nodes labeled by C and F is the same as the embedding type of the meet closure of the pair p, q. Thus there are $c, e, f \in T$ such that the meet closure of $\{c, e, f\}$ is a subtree of S with the weak embedding type sketched in Figure 4 with root node labeled $E = \{e\}$, leaf nodes labeled $C = \{c\}$ and $F = \{f\}$ such that $C : F \simeq P : Q$. Since $C : E \simeq A : B$ and $E : F \simeq A : B$, by the transitivity of $\mathcal{T}$, it follows that $(P, Q) \in \mathcal{T}$.

FIGURE 4

Case 4: There is $(A, B) = (\{a\}, \{b\}) \in \mathcal{T}$ with $|a| \neq |b|$:

We may assume that a and b are incomparable. Otherwise we can proceed as in the previous case.

Let $0 \leq i < j \leq d - 1$ be such that $(a \wedge b)\hat{\,} i \subseteq a$ and $(a \wedge b)\hat{\,} j \subseteq b$. By our assumption on Λ_F. Suppose first that $|a| < |b|$ and let $k = b(|a|)$.
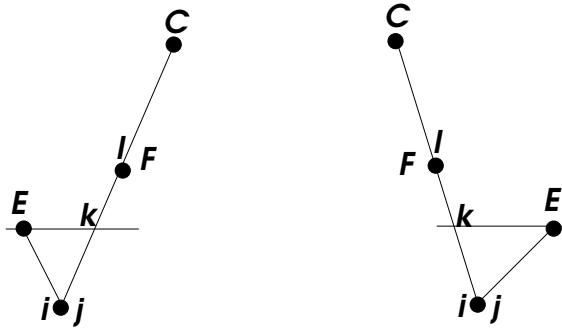

FIGURE 5

Let $(P, Q) = (\{p\}, \{q\}) \in \Lambda_F \setminus \{(\{\emptyset\}, \{\emptyset\})\}$ be such that $p \subsetneq q$ and $q(|p|) = l$. By our assumption on Λ_F such a (P, Q) must exist. Then there are $c, e, f \in T$ such that the meet closure of $\{c, e, f\}$ is a subtree of T with $f \subsetneq c$, $c(|f|) = l$, $|e| < |c|$, $f(|e|) = c(|e|) = k$, $e(|e \wedge f|) = i$ and $f(|e \wedge f|) = j$. Set $C = \{c\}$, $E = \{e\}$ and $F = \{f\}$. Then $C : F \simeq P : Q$. Since $C : E \simeq A : B$ and $E : F \simeq A : B$, it follows that $(P, Q) \in \mathcal{T}$. Thus by Case 3, $\mathcal{T} = \Lambda_F$. Similarly one treats the case $|b| < |a|$. The cases for $|a| < |b|$ and $|b| < |a|$ are illustrated in Figure 5.

To complete the proof, we check that one of the above cases must hold. If $\mathcal{T}$ is not a singleton, then it must contain some $(A, B) = (\{a\}, \{b\})$ with $a \neq b$. If $a \subsetneq b$ or $b \subsetneq a$, we are done by Case 3. If $|a| \neq |b|$, then we are done by Case 4. Otherwise, for all $(P, Q) = (\{p\}, \{q\}) \in \mathcal{T}$, we have $|p| = |q|$, and Case 2 applies. □

Now we prove the initial step $n = 1$ from the claim. If $\mathcal{T} = \{(\{\emptyset\}, \{\emptyset\})\}$, then $\mathcal{T} = \{(A, B) \in \Lambda_F : A \restriction (\{\emptyset\}, \emptyset) = B \restriction (\{\emptyset\}, \emptyset)\}$. If $\mathcal{T} = \{(\{p\}, \{q\}) \in \Lambda_F : |p| = |q|\}$, then $\mathcal{T} = \{(A, B) \in \Lambda_F : A \restriction (\emptyset, \{0\}) = B \restriction (\emptyset, \{0\})\}$. If $\mathcal{T} = \Lambda_F$, then $\mathcal{T} = \{(A, B) \in \Lambda_F : A \restriction (\emptyset, \emptyset) = B \restriction (\emptyset, \emptyset)\}$. This finishes the proof of the initial step $|F^*| = 1$.

The induction step at $n > 1$: Suppose now that the claim is true for all $F \in WEm^{<\omega}(T)$ with $1 \leq |F^*| < n$ and let $F \in WEm^{<\omega}(T)$ be such that $|F^*| = n$. Let $g = height(F^*)$. Note that $g > 1$. Let $\mathcal{T} \subseteq \Lambda_F$ be transitive for F, let (N, L) be a maximal node-level pair for all $\mathcal{T}$ and let $(C, D) \in \Lambda_F$ be such that $C \restriction (N, L) = D \restriction (N, L)$. Let $\Lambda_{F'}$ be a fixed minimal saturated set for $F' = F^* \setminus \{f\}$ where $f \in F$ is the right-most element of the top-level of F^* and let

> $\mathcal{T}' = \{(K, M) \in \Lambda_{F'} :$ there is $(A, B) \in \mathcal{T}$ such that $A^* \setminus \{a\} : B^* \setminus \{b\} \simeq K : M$, where a and b are the right-most elements of the top-levels of A^* and B^* respectively $\}$.

Claim 3.12.b. $\mathcal{T}'$ *is a transitive set.*

Proof of Claim. Let $(A'_1, B_1), (A'_2, B'_2) \in \mathcal{T}'$ and let $C', D', E' \in Em^{F'}(T)$ be such that $C' : D' \simeq A'_1 : B'_1$ and $D' : E' \simeq A'_2 : B'_2$. By the definition of the set $\mathcal{T}'$ there are $(A_1, B_1), (A_2, B_2) \in \mathcal{T}$ such that $A_1^* \setminus \{a_1\} : B_1^* \setminus \{b_1\} \simeq A'_1 : B'_1$ and $A_2^* \setminus \{a_2\} : B_2^* \setminus \{b_2\} \simeq A'_2 : B'_2$, where a_1, a_2, b_1 and b_2 are the right-most elements of the top-levels of A_1^*, A_2^*, B_1^* and B_2^* respectively. Without loss of generality we may assume that $C' : D' = A'_1 : B'_1$, $D' : E' = A'_2 : B'_2$, $A_1^* \setminus \{a_1\} : B_1^* \setminus \{b_1\} = A'_1 : B'_1$ and $A_2^* \setminus \{a_2\} : B_2^* \setminus \{b_2\} = A'_2 : B'_2$. Note that this means that the only difference between B_1 and A_2 can be the position of the right-most nodes b_1 and a_2 or the position of their meet. We will prove now that there is a pair $(K, M) \in \mathcal{T}$ such that $K : M \simeq B_1 : A_2$. Note that this will finish the proof of Claim 3.12.b. Suppose that there is a pair $(K, M) \in \mathcal{T}$ such that $K : M \simeq B_1 : A_2$. Then by the transitivity of $\mathcal{T}$ we have that there is a pair $(A_3, B_3) \in \mathcal{T}$ such that $A_1 : B_2 \simeq A_3 : B_3$.

This implies that $C' : E' \simeq A_3' : B_3'$, where $(A_3', B_3') \in \Lambda_{F'}$ is such that $A_3^* \setminus \{a_3\} : B_3^* \setminus \{b_3\} \simeq A_3' : B_3'$ and a_3 and b_3 are the right-most elements of the top-levels of A_3^* and B_3^* respectively. The proof follows by a case analysis.

Case 1: $f \in N$:

Then there is nothing to prove since $K = M = B_1 = A_2$.

Case 2: $f \notin N$:

Then either $g - 1 \in L$ or $g - 1 \notin L$.

Case 2.1: $g - 1 \in L$:

Note that in this case we must have $|a_2| = |b_1|$. Since $f \notin N$ and the node-level pair (N, L) is maximal such that $A \restriction (N, L) = B \restriction (N, L)$ for all $(A, B) \in \mathcal{T}$ there is a pair $(P, Q) \in \mathcal{T}$ such that $p \neq q$ where p and q are the right-most elements of the top-levels of P^* and Q^* respectively. Note that we must have $|p| = |q|$ since $g - 1 \in L$. Assume first that $p <_{lex} q$. Let $p' \in P^*(g - 2)$ and $q' \in Q^*(g - 2)$ be such that $p' \subseteq p$ and $q' \subseteq q$. Then either p' and q' are comparable or they are not comparable.

Case 2.1.1: p' and q' are comparable:

Then either $|p'| < |q'|$ or $|p'| = |q'|$ or $|p'| > |q'|$.

Case 2.1.1.1: $|p'| < |q'|$:

Then either $|p \wedge q| < |q'|$ or $|p \wedge q| \geq |q'|$.

Case 2.1.1.1.1: $|p \wedge q| < |q'|$:

Stretching the subtree $(P \cup Q)^*$ we can find a node q^1 (see Figure 6) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^1\}$ and $Q : (Q \setminus \{q\}) \cup \{q^1\} \simeq K : M$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.
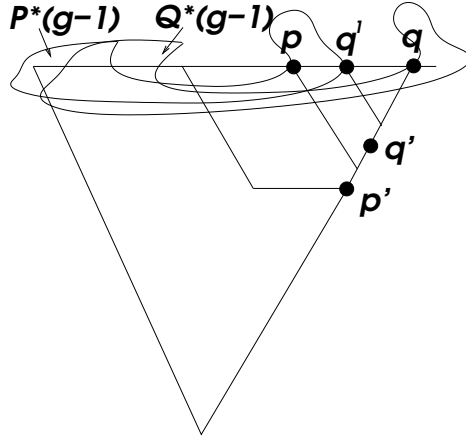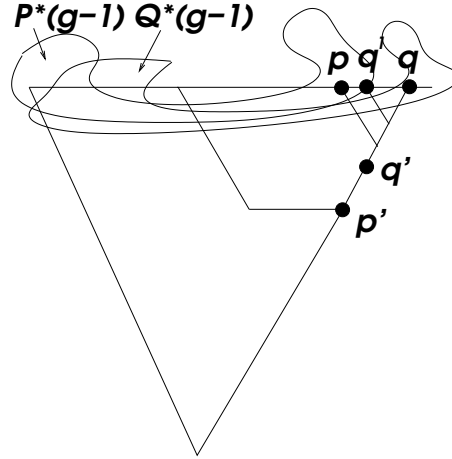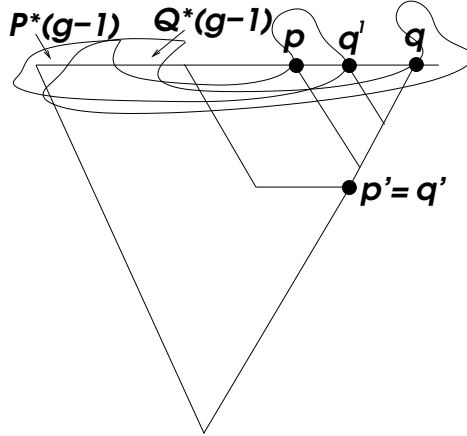

FIGURE 6

Case 2.1.1.1.2: $|p \wedge q| \geq |q'|$:

Stretching the subtree $(P \cup Q)^*$ we can find a node q^1 (see Figure 7) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^1\}$ and

$Q : (Q \setminus \{q\}) \cup \{q^1\} \simeq K : M$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.

FIGURE 7

Case 2.1.1.2: $|p'| = |q'|$:

Stretching the subtree $(P \cup Q)^*$ we can find a node q^1 (see Figure 8) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^1\}$ and $Q : (Q \setminus \{q\}) \cup \{q^1\} \simeq K : M$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.
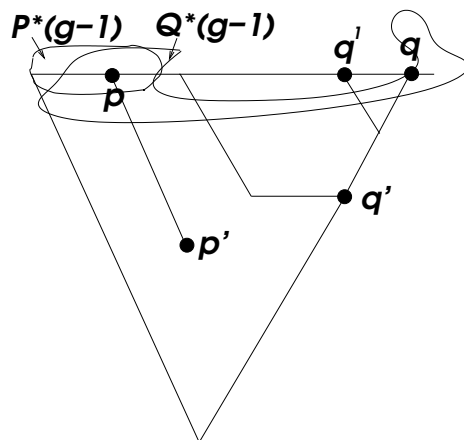

FIGURE 8

Case 2.1.1.3: $|p'| > |q'|$:

Similar to Case 2.1.1.1.

Case 2.1.2: p' and q' are not comparable:

Then either $p' <_{lex} q'$ or $q' <_{lex} p'$.

Case 2.1.2.1: $p' <_{lex} q'$:

Then either $|p'| < |q'|$ or $|p'| = |q'|$ or $|p'| > |q'|$.

Case 2.1.2.1.1: $|p'| < |q'|$:

Stretching the subtree $(P \cup Q)^*$ we can find a node q^1 (see Figure 9) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^1\}$ and $Q : (Q \setminus \{q\}) \cup \{q^1\} \simeq K : M$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.
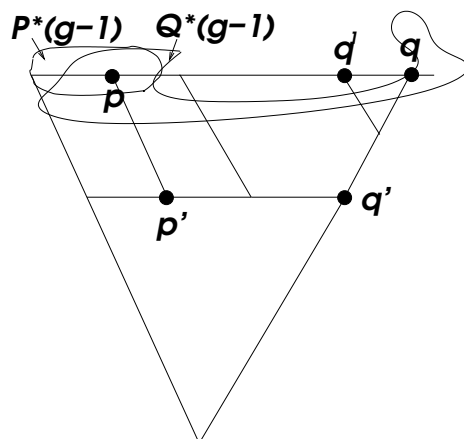

FIGURE 9

Case 2.1.2.1.2: $|p'| = |q'|$:

Stretching the subtree $(P \cup Q)^*$ we can find a node q^1 (see Figure 10) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^1\}$ and $Q : (Q \setminus \{q\}) \cup \{q^1\} \simeq K : M$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.

FIGURE 10

Case 2.1.2.1.3: $|p'| > |q'|$:

Stretching the subtree $(P \cup Q)^*$ we can find a node q^1 (see Figure 11) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^1\}$ and $Q : (Q \setminus \{q\}) \cup \{q^1\} \simeq K : M$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.
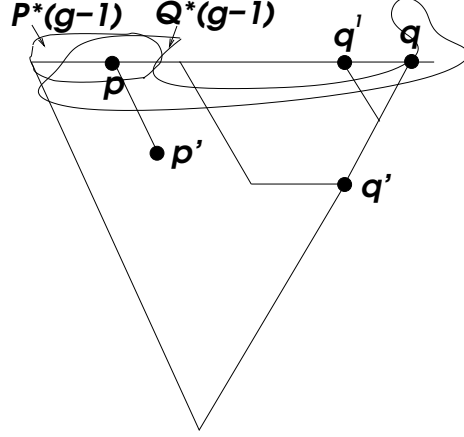

FIGURE 11

Let us recall that we have started the proof of Case 2.1 with the assumption that $p <_{lex} q$. It should be clear that the same proof works in the case $q <_{lex} p$ with p instead of q.

Case 2.2: $g - 1 \notin L$:

Note that in this case the length of a_2 and b_1 can be different. Since $g - 1 \notin N$ and the node-level pair (N, L) is maximal such that $A \upharpoonright (N, L) = B \upharpoonright (N, L)$ for all $(A, B) \in \mathcal{T}$ there is a pair $(P, Q) \in \mathcal{T}$ such that $|p| \neq |q|$ where p and q are the right-most elements of the top-levels of P^* and Q^* respectively. Assume first that $|p| < |q|$. Then either $|F^*(g - 1)| = 1$ or $|F^*(g - 1)| > 1$.

Case 2.2.1: $|F^*(g - 1)| = 1$:

Let $p' \in P^*(g - 2)$ and $q' \in Q^*(g - 2)$ be such that $p' \subseteq p$ and $q' \subseteq q$. Then either p' and q' are comparable or they are not comparable.

Case 2.2.1.1: p' and q' are comparable:

Then either $|p'| < |q'|$ or $|p'| = |q'|$ or $|p'| > |q'|$.

Case 2.2.1.1.1: $|p'| < |q'|$:

Then either p and q are comparable or they are not comparable.

Case 2.2.1.1.1.1: p and q are comparable:

Assume first that $|q'| < |p|$. Stretching the subtree $(P \cup Q)^*$ we can find nodes q^1, q^2, q^3 and q^4 (see Figure 12) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^i\}$ for every $1 \le i \le 4$ and $Q : (Q \setminus \{q\}) \cup \{q^i\} \simeq K : M$ for some

i, $1 \leq i \leq 4$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.
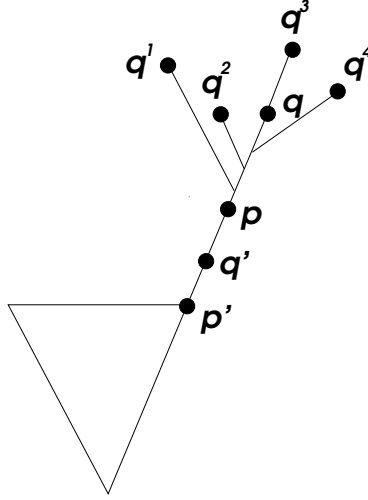

FIGURE 12

It should be clear that the same proof works in the case $|q'| \geq |p|$.

Case 2.2.1.1.1.2: p and q are not comparable:

Assume first that $|q'| < |p \wedge q|$ and $p <_{lex} q$. Stretching the subtree $(P \cup Q)^*$ we can find nodes q^1, q^2, q^3 and q^4 (see Figure 13) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^i\}$ for every $1 \leq i \leq 4$ and $Q : (Q \setminus \{q\}) \cup \{q^i\} \simeq K : M$ for some i, $1 \leq i \leq 4$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.

It should be clear that the same proof works in the case $|q'| \geq |p \wedge q|$ or $q <_{lex} p$.

Case 2.2.1.1.2: $|p'| = |q'|$:

Similar to Case 2.2.1.1.1.

Case 2.2.1.1.3: $|p'| > |q'|$:

Similar to Case 2.2.1.1.1.

Case 2.2.1.2: p' and q' are not comparable:

Then either $|p'| < |q'|$ or $|p'| = |q'|$ or $|p'| > |q'|$.

Case 2.2.1.2.1: $|p'| < |q'|$:

Assume first that $|q'| < |p|$ and $p' <_{lex} q'$. Stretching the subtree $(P \cup Q)^*$ we can find nodes q^1, q^2, q^3 and q^4 (see Figure 14) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^i\}$ for every $1 \leq i \leq 4$ and $Q : (Q \setminus \{q\}) \cup \{q^i\} \simeq K : M$ for some i, $1 \leq i \leq 4$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.
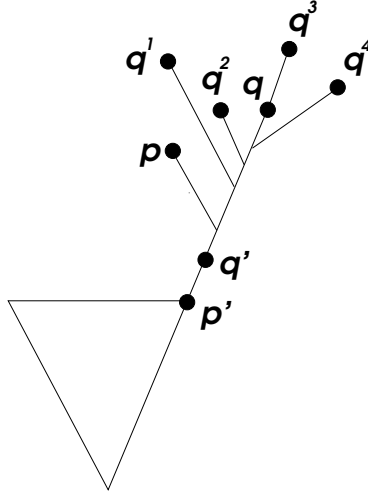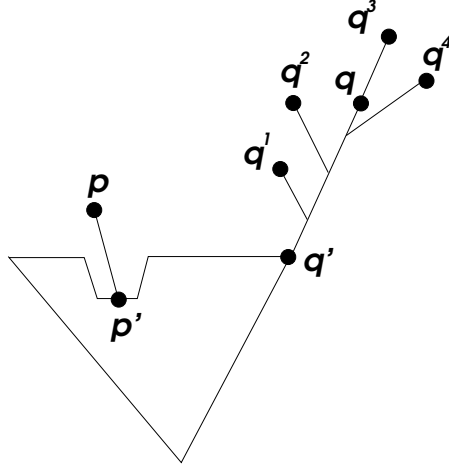
FIGURE 13

FIGURE 14

It should be clear that the same proof works in the case $|q'| \geq |p|$ or $q <_{lex} p$.

Case 2.2.1.2.2: $|p'| = |q'|$:
Similar to case 2.2.1.2.1.

Case 2.2.1.2.3: $|p'| = |q'|$:
Similar to case 2.2.1.2.1

Case 2.2.2: $|F^*(g-1)| > 1$:
Let $p' \in P^*(g-2)$ and $q' \in Q^*(g-2)$ be such that $p' \subseteq p$ and $q' \subseteq q$. Then either p' and q' are comparable or they are not comparable.

Case 2.2.2.1: p' and q' are comparable:

Then either $|p'| < |q'|$ or $|p'| = |q'|$ or $|p'| > |q'|$.

Case 2.2.2.1.1: $|p'| < |q'|$:

Then either p and q are comparable or they are not comparable.

Case 2.2.2.1.1.1: p and q are comparable:

Assume first that $|p| < |q'|$. Stretching the subtree $(P \cup Q)^*$ we can find a node q^1 (see Figure 15) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^1\}$ and $Q : (Q \setminus \{q\}) \cup \{q^1\} \simeq K : M$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.
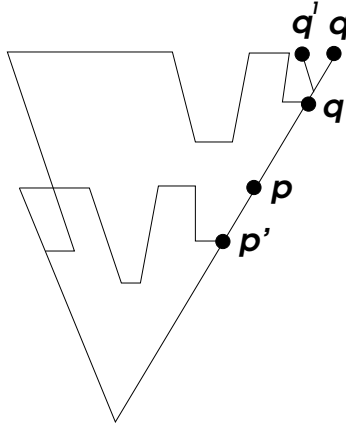

FIGURE 15

It should be clear that the same proof works in the case $|p| \geq |q'|$.

Case 2.2.2.1.1.2: p and q are not comparable:

Assume first that $|q'| < |p \wedge q|$ and $p <_{lex} q$. Stretching the subtree $(P \cup Q)^*$ we can find a node q^1 (see Figure 16) such that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^1\}$ and $Q : (Q \setminus \{q\}) \cup \{q^1\} \simeq K : M$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.

It should be clear that the same proof works in the case $|q'| \geq |p \wedge q|$ or $q <_{lex} p$.

Case 2.2.2.1.2: $|p'| = |q'|$:

Similar to Case 2.2.2.1.1.

Case 2.2.2.1.3: $|p'| > |q'|$:

Similar to Case 2.2.2.1.1.

Case 2.2.2.2: p' and q' are not comparable:

Then either $|p'| < |q'|$ or $|p'| = |q'|$ or $|p'| > |q'|$.

Case 2.2.2.2.1: $|p'| < |q'|$:

Assume first that $|p| < |q'|$ and $p' <_{lex} q'$. Stretching the subtree $(P \cup Q)^*$ we can find a node q^1 (see Figure 17) such
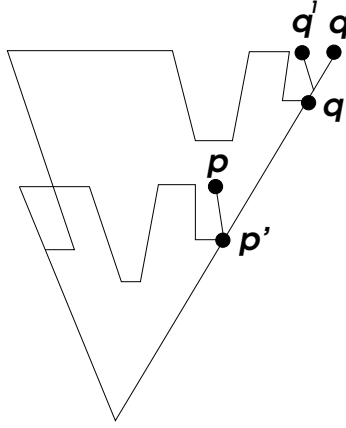
FIGURE 16

that $P : Q \simeq P : (Q \setminus \{q\}) \cup \{q^1\}$ and $Q : (Q \setminus \{q\}) \cup \{q^1\} \simeq K : M$. By the transitivity of $\mathcal{T}$ we have that $(K, M) \in \mathcal{T}$.
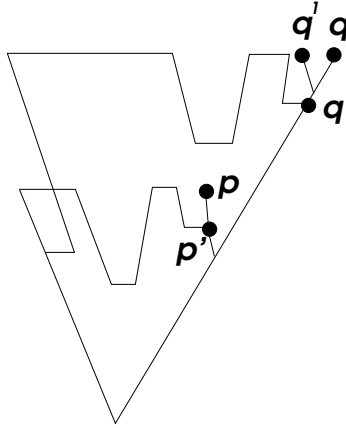

FIGURE 17

It should be clear that the same proof works in the case $|p| \geq |q'|$ or $q <_{lex} p$.

Case 2.2.2.2.2: $|p'| = |q'|$:

Similar to case 2.2.2.2.1.

Case 2.2.2.2.3: $|p'| = |q'|$:

Similar to case 2.2.2.2.1.

Let us recall that we have started the proof of Case 2.2 with the assumption that $|p| < |q|$. It should be clear that the same proof works in the case $|q| < |p|$ with p instead of q.

□

Define a node-level pair (N', L') of F' as follows: set $N' = N \setminus \{f\}$, and let $L' = L \setminus \{g - 1\}$ if $|F^*(g - 1)| = 1$, where $height(F^*) = g$ and $L' = L$ otherwise. Note that the node-level pair (N', L') is a maximal node-level pair for $\mathcal{T}'$. Let $(C', D') \in \Lambda_{F'}$ be such that $C' : D' = C^* \setminus \{c\} : D^* \setminus \{d\}$, where c and d are the right-most elements of the top-levels of C^* and D^* respectively. By the inductive hypothesis applied to F' and $\mathcal{T}'$, $(C', D') \in \mathcal{T}'$. Let $(K, M) \in \mathcal{T}$ be such that $K^* \setminus \{k\} : M^* \setminus \{m\} = C' : D' = C^* \setminus \{c\} : D^* \setminus \{d\}$, where k and m are the right-most elements of the top-levels of K^* and M^* respectively. We will prove now that there are pairs $(A_1, B_1), (A_2, B_2) \in \mathcal{T}$ such that $A_1 : B_1 \simeq K : C$ and $A_2 : B_2 \simeq M : D$. Note that this will finish the proof of the theorem since by the transitivity of $\mathcal{T}$ applied twice we have that $(C, D) \in \mathcal{T}$.

Claim 3.12.c. *There are pairs* $(A_1, B_1), (A_2, B_2) \in \mathcal{T}$ *such that* $A_1 : B_1 \simeq K : C$ *and* $A_2 : B_2 \simeq M : D$.

Proof of Claim 3.12.c. The proof proceeds by the same case analysis as in Claim 3.12.b.   □

□

By Theorem 3.12 and Theorem 3.6 we get the following result (see also [2]).

Theorem 3.13. *Let S be a strongly embedded subtree of T of height ω and $F \in WEm^{<\omega}(S)$. Then for every equivalence relation $\mathbb{E}$ on $Em^F(S)$ there is a strongly embedded subtree U of S of height ω and there is a node-level pair (N, L) of F such that for every $A, B \in Em^F(U)$, we have*

$$A \mathbb{E} B \iff A \restriction (N, L) = B \restriction (N, L).$$

Proof. Let $\mathbb{E}$ be an equivalence relation on $Em^F(S)$. By Theorem 3.6 there is a strongly embedded subtree U of S of height ω and there is $\mathcal{T} \subseteq \Lambda_F$ transitive for U such that for every $A, B \in Em^F(U)$, we have

$$A \mathbb{E} B \iff A \mathbb{E}_{\mathcal{T}} B.$$

By Theorem 3.12 there is a node-level pair (N, L) of F such that

$$\mathcal{T} = \{(A, B) \in \Lambda_F : A \restriction (N, L) = B \restriction (N, L).$$

Thus, for every $A, B \in Em^F(U)$

$$A \mathbb{E} B \iff A \mathbb{E}_{\mathcal{T}} B \iff A \restriction (N, L) = B \restriction (N, L) .$$

This finishes the proof of the theorem.   □

In the case of colorings of nodes of T we get the following nice result discovered independently by the author and by Milliken [2].

Corollary 3.14. *Let $f : T \to \mathbb{N}$ be arbitrary. Then there is a strongly embedded subtree S of T of height ω such that exactly one of the following three alternatives holds:*

(1) $\forall s, t \in S \ f(s) = f(t) \ \Leftrightarrow \ s = t,$
(2) $\forall s, t \in S \ f(s) = f(t) \ \Leftrightarrow \ \forall n(s \in S(n) \ \Leftrightarrow \ t \in S(n)) \ or$
(3) $\forall s, t \in S \ f(s) = f(t).$

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Faculty of Information Technology,
Trg Republike 3, 11000 Belgrade, Serbia
E-mail address: `vojkan.vuksanovic@fit.edu.yu`