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POLYTOPES DERIVED FROM SPORADIC SIMPLE GROUPS

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ABSTRACT. In this article, certain of the sporadic simple groups are analysed, and the polytopes having these groups as automorphism groups are characterised. The sporadic groups considered include all with order less than 4030387201, that is, all up to and including the order of the Held group. Four of these simple groups yield no polytopes, and the highest ranked polytopes are four rank 5 polytopes each from the Higman-Sims group, and the Mathieu group M_{24}.

1. INTRODUCTION

The finite simple groups are the building blocks of finite group theory. Most fall into a few infinite families of groups, but there are 26 (or 27 if the Tits group ${}^2F_4(2)'$ is counted also) which these infinite families do not include. These sporadic simple groups range in size from the Mathieu group M_{11} of order 7920, to the Monster group M of order approximately 8×10^{53}. One key to the study of these groups is identifying some geometric structure on which they act. This provides intuitive insight into the structure of the group. Abstract regular polytopes are combinatorial structures that have their roots deep in geometry, and so potentially also lend themselves to this purpose. Furthermore, if a polytope is found that has a sporadic group as its automorphism group, this gives (in theory) a presentation of the sporadic group over some generating involutions.

In [6], a simple algorithm was given to find all polytopes acted on by a given abstract group. In this article, that algorithm is applied to various sporadic simple groups, and the polytopes for each group are enumerated. For certain of the groups, this confirms results obtained previously. For others, the results are new. In particular, this article gives new polytopes acted on by the Mathieu group M_{24}, the Higman-Sims group HS, the Janko group J_3, the McLaughlin group McL and the Held group He.

The algorithm, as stated in [6], is as follows.

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- Let S be the set of all involutions of Γ, and let T be a set consisting of one representative of each orbit of the action of $K = \mathrm{Aut}(\Gamma)$ on S. Let S_1 be a set of 1-tuples corresponding to the elements of T.
- Given S_k, a collection of k-tuples of elements of S, construct S_{k+1} as follows. For each tuple t in S_k, let K_t be the stabilizer of t in K. Let T be a set consisting of one representative of each orbit of the action of K_t on S. For each $u \in T$, if appending u to t yields a set of involutions that has a string diagram and satisfies the intersection property, add it to P_{k+1} or S_{k+1} respectively as these involutions generate Γ or a proper subgroup of Γ.
- Stop when some S_n is empty.
- Applying the coset poset construction to each $t \in P_k$ yields the polytopes of rank k whose automorphism groups are Γ.

A naive application of this algorithm works well for smaller groups. For larger groups (in particular J_3 and He), even storing the set S is too much for a modern computer's memory. The changes needed to make the algorithm work for these groups is explained in Section 6.

The canonical reference for abstract regular polytopes is [18]. For convenience, the basic theory is outlined here. An abstract n-polytope is a partially ordered set $\mathcal{P}$ with unique minimal and maximal elements, and an order-preserving rank function from $\mathcal{P}$ onto $\{-1, 0, \ldots, n-1, n\}$. This poset corresponds to the face-lattice of the polytope in the classical theory. The *flags* of $\mathcal{P}$ are the maximal totally ordered subsets of $\mathcal{P}$, which must have size $n+1$. For any flag Φ and each $i \in \{0, \ldots, n-1\}$, we require that there exist a unique flag Φ' of $\mathcal{P}$ differing from Φ only by an element of rank i. This allows the definition of *exchange maps* ϕ_i that map each Φ to the corresponding Φ'. A further condition on an abstract polytope is that the group generated by the exchange maps acts transitively on the set of flags of the polytope. The term "exchange maps" is due to Gordon Williams (see [20]).

Another action on the flags of the polytope may be defined, that is, the action of the polytope's automorphism group. The polytope is called *regular* if this action too is transitive on the set of flags. A group generated by involutions (a *ggi*) is a group W with a specified generating set $\langle s_0, \ldots, s_{n-1}\rangle$ of involutions. If the group has a string diagram, that is, if $(s_i s_j)^2 = 1$ whenever $i \neq j, j \pm 1$, it is called a *string* ggi, or *sggi* for short. If it also satisfies the *intersection property*, that is, $\langle s_i : i \in I\rangle \cap \langle s_i : i \in J\rangle = \langle s_i : i \in I \cap J\rangle$, it is called a string *C-group*. String Coxeter groups are examples of string C-groups. The most important result regarding abstract regular polytopes is that there is a one-to-one correspondence between regular polytopes and string C-groups. See Chapter 2 of [18], in particular Corollary 2E13.

This correspondence is not complex. Let W be a string C-group, and let $H_i = \langle s_j : j \neq i\rangle$. A unique polytope $\mathcal{P}(W)$ is obtained by letting $\mathcal{P}(W)$ be the poset $\{P_{-1}, P_n\} \cup \{uH_i : u \in W, 0 \leq i \leq n-1\}$, with unique maximal and minimal elements P_n and P_{-1}, and with $uH_i \leq vH_j$ if and only if $i \leq j$ and $uH_i \cap vH_j$ is nonempty. The polytope $\mathcal{P}(W)$ is regular, and its automorphism group is isomorphic to W. Conversely, if $\mathcal{P}$ is a regular polytope, let W be its automorphism group. Choose and fix one flag Ψ of $\mathcal{P}$, and for $0 \leq i \leq n-1$, let s_i be the element of W mapping Ψ to $\Psi\phi_i$. Then the s_i generate W and make it a string C-group. Furthermore, the polytope constructed from W will be isomorphic to $\mathcal{P}$. Writing $W = \Gamma$, and $\mathrm{Aut}(\mathcal{P}) = \Gamma(\mathcal{P})$ (as some authors do), the relationship between regular polytopes and string C-groups may be written succintly as $\mathcal{P}(\Gamma(\mathcal{P})) \cong \mathcal{P}$ and $\Gamma(\mathcal{P}(\Gamma)) \cong \Gamma$.

It follows that if a simple group does act on a polytope as its automorphism group, the group's structure is truly bound up in the structure of the polytope.

Before moving on, a few more basic concepts will be outlined. A *section* of a polytope is a subset of the form $F/G = \{x : G \leq x \leq F\}$. The sections of a polytope are also polytopes. The *faces* are sections of the form F/P_{-1}, and the *cofaces* those of the form P_n/G. The *rank* of a polytope is n, where the size of the flags is $n+1$, and the *rank* of an element F of a polytope is the rank of the corresponding face. If, for sections F/G of rank 2, the isomorphism type of the section depends only on the ranks of F and G and not on F and G themselves, then $\mathcal{P}$ is said to be *equivelar*, and has a well-defined *Schläfli symbol* $\{p_1, \ldots, p_{n-1}\}$ where F/G is a p_i-gon whenever $\operatorname{rank} F - 2 = i = \operatorname{rank} G + 1$. Note that all regular polytopes are equivelar. The rank $n-1$ faces and cofaces are called the *facets* and *vertex figures* respectively.

A structure reversing bijection from a polytope $\mathcal{P}$ to a polytope $\mathcal{Q}$ is called a *duality*, with $\mathcal{P}$ being the *dual* of $\mathcal{Q}$ (and vice-versa). If in fact $\mathcal{P} \cong \mathcal{Q}$, then $\mathcal{P}$ is said to be *self-dual*.

If there exists a structure-preserving *sur*jection from a polytope $\mathcal{P}$ to another polytope $\mathcal{Q}$, then $\mathcal{P}$ is said to *cover* (or *be a cover for* $\mathcal{Q}$), and $\mathcal{Q}$ is a *quotient* of $\mathcal{P}$. Every polytope is covered by a regular polytope (see [5]). Furthermore, if there exists a polytope with certain regular facets and vertex figures, then there exists a *universal* such polytope which covers all others (see Theorem 4A2 of [18], and Theorem 2.5 of [8]).

As demonstrated earlier, a regular polytope is isomorphic to a *coset geometry* of its automorphism group, using the cosets $H_i = \langle s_j : j \neq i \rangle$. For this reason, the polytopes for many smaller simple groups have been classified as the geometries for the groups were classified. It should be noted, however, that most such classifications fail to distinguish polytopes from their duals, since a polytope and its dual are regarded as identical geometries. More importantly, if a classification of geometries for a particular group is restricted to subcategory of geometries that does not encapsulate polytopes, it will fail to completely classify all polytopes related to the group.

In particular, in [1], [2], [12], [13], [14], [15], and [16] the residually weakly primitive geometries of various sporadic simple groups were analysed. Although that work had the potential to discover some polytopes for the groups studied, they could not have classified these polytopes. In point of fact, they did not discover polytopes for these groups.

In [17] (originally appearing as [19]), most almost simple groups of order 10^6 or less were studied, and the polytopes for each such group were enumerated. In particular, [17] completely enumerated the abstract polytopes whose automorphism groups are the sporadic simple groups M_{11}, M_{12}, M_{22}, J_1 and J_2.

The main contribution of this article is therefore threefold:

- It classifies the polytopes related to the Mathieu groups M_{23} and M_{24}, the Tits group, the Higman-Sims group, the third Janko group J_3, the McLaughlin group and the Held group.
- It confirms the results of [17].
- It also serves to distinguish the polytopes discovered in [17] from their duals.

The structure of this article is simple. Section 2 discusses some of the more interesting polytopes that arise from the previously published work. Section 3 desribes the polytopes arising from the Mathieu group M_{23}, Section 4 describes the polytopes for the Tits group ${}^2F_4(2)'$, Section 5 for the Higman-Sims, Section 6 for J_3, Section 7 for M_{24} Section 8 for the McLaughlin group McL, and Section 9 for the Held group He. This article provides some summary information for the polytopes discovered. For example, Table 1 summarises the numbers of polytopes for each group, with the numbers of self-dual polytopes indicated in brackets (unless obvious). Far more detailed information is given in the auxiliary information for this article, [7], including (amongst other things) generating sets for the automorphism groups of every polytopes discovered.

2. Interesting Polytopes

This section describes the polytopes arising from the sporadic simple groups examined in [17].

The Mathieu groups M_{11} and M_{22} are not automorphism groups of any polytopes. M_{12} is more interesting. It has 40 rank 3 polytopes, which are summarised in Table 2. The entry in a row labeled p and a column labeled q is then number of polytopes of type $\{p, q\}$ discovered. This format also is adopted for similar tables elsewhere in the article.

Six of the rank 3 polytopes found are self-dual. M_{12} has one self-dual rank 4 polytope, of type $\{6, 3, 6\}$, a finite quotient of the infinite universal $\{\{6, 3\}_{(4,0)}, \{3, 6\}_{(4,0)}\} = \{\{6, 3\}_8, \{3, 6\}_8\}$.

The remaining rank 4 polytopes (excluding duals) of M_{12} are of types $\{\{3, 5\}_5, \{5, 4|5\}\}$, $\{3, 6, 4\}$ (facet $\{3, 6\}_{(3,0)}$), $\{3, 6, 6\}$ (facet $\{3, 6\}_8$), $\{3, 8, 4\}$, $\{3, 10, 4\}$, $\{4, 5, 5\}$ (two, both with vertex figures $\{5, 5\}_5$, one with facets

TABLE 1. Polytopes for Sporadic Groups.

Group	Order	Rank			
		$=3$	$=4$	$=5$	≥ 6
M_{11}	7920	0	0	0	0
M_{12}	95040	40 (6)	27 (1)	0	0
J_1	175560	296(0)	4(0)	0	0
M_{22}	443520	0	0	0	0
$J_2 = HJ$	604800	261(13)	31(3)	0	0
M_{23}	10200960	0	0	0	0
${}^2F_4(2)'$	17971200	468(20)	57(5)	0	0
HS	44352000	465(39)	111(3)	4(0)	0
J_3	50232960	584(22)	2(2)	0	0
M_{24}	244823040	946(34)	310(0)	4(0)	0
McL	898128000	0	0	0	0
He	4030387200	2292(84)	145(7)	0	0

TABLE 2. Rank 3 polytopes for the Mathieu group M_{12}.

	5	6	8	10
5	0	1	1	0
6	1	3(1)	6	2
8	1	6	6(4)	5
10	0	2	5	1(1)

$\{4,5|5\}$), $\{4,5,6\}$ (facets $\{4,5\}_6$), $\{4,6,5\}$ (two examples), $\{4,6,6\}$ (facets $\{4,6\}_5$), $\{4,8,4\}$ and $\{\{5,3\}_5,\{3,6\}_8\}$.

The rank 4 polytopes of J_1 have been studied in some detail. There are two dual pairs. One has type $\{5,3,5\}$ with dodecahedral facets and hemi-icosahedral vertex figures (and its dual). It became important in the construction of the universal polytope of this type, which has group $J_1 \times L_2(19)$. See [9] for more information. The other dual pair has type $\{5,6,5\}$, and is a pair of universal polytopes, that is, they have no proper cover with the same facet and vertex figure types. The vertex figures (of one of the dual pair) are the unique map $\mathcal{L}$ of type $\{6,5\}$ with group of order 660. The rank 3 polytopes of J_1 are summarised in Table 3. In [10] it is shown how the polytope of type $\{5,3,5\}$ may be used as a basis to construct all of the rank 4 thin residually connected geometries of the Janko group J_1.

The second Janko group J_2 is the automorphism group of 296 rank 3 polytopes, summarised in Table 4, and of 31 rank 4 polytopes. Of the

TABLE 3. Rank 3 polytopes for the Janko group J_1.

	3	5	6	7	10	11	15	19
3	0	0	0	1	2	1	2	4
5	0	2(0)	0	2	4	0	4	8
6	0	0	0	1	2	3	4	6
7	1	2	1	2(0)	4	3	8	9
10	2	4	2	4	4(0)	6	8	14
11	1	0	3	3	6	2(0)	6	9
15	2	4	4	8	8	6	6(0)	18
19	4	8	6	9	14	9	18	22(0)

TABLE 4. Rank 3 polytopes for the Janko group J_2.

	3	4	5	6	7	8	10	12	15
3	0	0	0	0	0	0	0	1	1
4	0	0	0	0	1	0	0	0	2
5	0	0	0	3	3	2	6	1	3
6	0	0	3	3(1)	3	3	12	4	6
7	0	1	3	3	2(2)	2	9	2	6
8	0	0	2	3	2	3(1)	8	0	4
10	0	0	6	12	9	8	30(2)	8	13
12	1	0	1	4	2	0	8	1(1)	3
15	1	2	3	6	6	4	13	3	10(6)

rank 4 polytopes, 3 are self-dual, of type $\{5,6,5\}$, $\{\{6,3\}_{10},\{3,6\}_{10}\}$ and $\{\{10,3\}_6,\{3,10\}_6\}$. The remaining rank 4 polytopes are of type $\{3,5,5\}$, $\{3,8,3\}$, $\{3,8,4\}$, $\{3,8,5\}$ (three examples), $\{\{5,3\}_5,\{3,6\}_{10}\}$, $\{4,8,5\}$ (two examples), $\{5,5,10\}$, $\{5,8,5\}$ (two examples), $\{6,3,10\}$ and their duals.

The remaining sections consider one by one the sporadic simple groups not covered in [17], in order of increasing size, and detail the polytopes of which they act automorphically.

3. Mathieu group M_{23} of order $10200960 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$

There are no polytopes which have the Mathieu group M_{23} as their automorphism group.

4. Tits group ${}^2F_4(2)'$ of order $17971200 = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$

The Tits group is the automorphism group of 468 rank 3 polytopes and 57 rank 4 polytopes. The rank 3 polytopes are summarised in Table 5.

Amongst the rank 4 polytopes are 5 self-dual polytopes, and 26 dual pairs. The self-dual polytopes have facets of type $\{4,4\}_{(5,0)}$ with group of order 200, $\{5,5\}$ with group of order 720, $\{4,6\}$ of order 1440 (a quotient of $\{4,6\}_8$), $\{5,5\}$ of order 5120, and $\{8,6\}$ of order 11232. For the first three of these, this description gives enough information to completely identify the facet (using [6] for example). Their Atlas Canonical Names (see the web version of [6]) are $\{4,4\}*200$, $\{5,5\}*720$ and $\{4,6\}*1440a$.

TABLE 5. Rank 3 polytopes for the Tits group $^2F_4(2)'$.

	3	4	5	6	8	10	12	13
3	0	0	0	0	0	1	0	1
4	0	0	0	0	4	5	5	6
5	0	0	0	2	4	2	8	7
6	0	0	2	0	6	6	10	12
8	0	4	4	6	22(4)	24	18	22
10	1	5	2	6	24	21(5)	12	12
12	0	5	8	10	18	12	27(5)	22
13	1	6	7	12	22	12	22	20(6)

5. Higman-Sims group *HS* of order $44352000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$

The Higman-Sims group acts on 465 rank 3 polytopes, 111 rank 4 polytopes, and four rank 5 polytopes. The rank 3 polytopes are summarised in Table 6.

Of the 111 rank 4 polytopes of the Higman-Sims group, only three are self-dual. The facets of these are:

TABLE 6. Rank 3 polytopes for the Higman-Sims group.

	3	4	5	6	7	8	10	12	15
3	0	0	0	0	0	0	0	0	1
4	0	0	0	0	2	0	2	0	7
5	0	0	0	0	2	4	0	2	2
6	0	0	0	2(2)	5	11	7	6	6
7	0	2	2	5	9(5)	15	11	5	11
8	0	0	4	11	15	31(9)	24	16	23
10	0	2	0	7	11	24	10(6)	10	14
12	0	0	2	6	5	16	10	7(3)	8
15	1	7	2	6	11	23	14	8	18(14)

- type $\{8,3\}$, a quotient of $\{8,3\}_8$, with group of order 336; that is, the polytope with Atlas Canonical Name $\{8,3\} * 336a$,
- type $\{5,5\}$, self-dual, with group of order 600; that is, $\{5,5\} * 600$, and
- type $\{4,5\}$, with group of order 1920; that is, $\{4,5\} * 1920$.

The rank 5 polytopes of the Higman-Sims group form two dual pairs, of types $\{3,8,5,3\}$ and $\{5,8,5,3\}$ and their duals. The vertex figures of these are identical, their group is a non-solvable non-simple group of order 7680, and centre of order 2. The group $\{s'_0, s'_1, s'_2, s'_3, s'_4\}$ of the $\{5,8,5,3\}$ may be constructed from the group $\{s_0, s_1, s_2, s_3, s_4\}$ of the $\{3,8,5,3\}$ by allowing $s'_i = s_i$ for $i \neq 1$, and $s'_i = s_i\omega$ where ω is the generator of the centre of $\langle s_1, s_2, s_3, s_4\rangle$. The facets of these polytopes have groups of order 252000, the latter being $P\Sigma U_3(5)$, that is, the index 3 subgroup of the automorphism group of the Unitary group $U_3(5)$. In fact, there are two polytopes of type $\{3,8,5\}$ which have $P\Sigma U_3(5)$ as their automorphism group. The one in question here is the one whose facets $\{3,8\}$ have groups of order 720; that is, the facets are $\{3,8\} * 720$.

6. Janko group J_3 of order $50232960 = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$

As mentioned earlier, the algorithm given in [6] could not be directly applied to the larger groups considered here, since it requires too much information to be stored. Two changes were introduced to enable the polytopes of J_3 and *He* to be found, one in the way involutions were stored, the other in the way the algorithm orders its operations.

When GAP is asked to compute the right transversal of a group, it stores it in a compact manner that requires much less memory than would the list of elements of the right transversal. The involutions of J_3 may be stored as the index of elements of the right transversal of the normaliser in J_3 of a single involution. It is easy (and efficient) to convert an index to an involution. The algorithm does not need to convert involutions to indexes, although if it did, this, too, could be made efficient. The automorphism group of J_3 (that is, J_3 itself) can be represented as as a group permuting these indices, which in turn means that the stabilisers needed by the algorithm also require less memory.

For the Held group, there are two conjugacy classes of involutions, so there are two right transversals that need to be indexed.

The second change to the way the algorithm works is to ensure it operates in a "depth first" rather than a "breadth first" manner. As stated in [6], the algorithm will begin by finding S_1, then find the whole of S_2 before calculating S_3, then S_4 and so on. For large groups, these sets can become extremely large, and overwhelm the computer's memory before the algorithm can complete. To overcome this problem, each S_k is computed only partially, and then a corresponding part of S_{k+1} is analysed before the next part of S_k is computed.

So, to analyse $t = (s_1, \ldots, s_k) \in S_k$, we proceed as follows.

- Find the stabiliser K_t of t in $K = \mathrm{Aut}(\Gamma)$.
- For each orbit Ω of the action of K_t on S, find a representative $u \in \Omega$.
- If appending u to t does **not** yield a set of involutions that has a string diagram and satisfies the intersection property, then move to the next orbit.
- Otherwise, if $\{s_1, \ldots, s_k, u\}$ generates Γ, output $(s_1, \ldots, s_k, u)$ and move to the next orbit.
- If $\{s_1, \ldots, s_k, u\}$ generates a proper subgroup of Γ, then analyse $(s_1, \ldots, s_k, u)$ recursively.
- When all orbits Ω have been tested, the analysis of t is complete.

All polytopes for a given group may therefore be found by analysing in turn each element (s_1) of S_1.

This method was applied to the Janko group J_3 and to the Held group *He*. The results for J_3 are summarised below, and for *He* may be found in Section 9.

The third Janko group J_3 acts on 584 rank 3 polytopes, and only two rank 4 polytopes. The two rank 4 polytopes are not a dual pair, but are self-dual of types $\{5, 9, 5\}$ and $\{3, 17, 3\}$. The facets of these are polytopes whose automorphism groups are the projective special linear groups $L_2(19)$ of order 3420 and $L_2(16)$ of order 4080 respectively. The rank 3 polytopes are summarised in Table 7.

TABLE 7. Rank 3 polytopes for the Janko group J_3.

	3	4	5	6	8	9	10	12	15	17
3	0	0	0	0	0	0	1	2	2	2
4	0	0	0	0	0	2	1	0	1	0
5	0	0	0	1	4	13	6	5	8	8
6	0	0	1	1(1)	2	8	1	2	4	1
8	0	0	4	2	0	11	6	2	8	5
9	0	2	13	8	11	29(7)	18	16	33	23
10	1	1	6	1	6	18	9(3)	5	12	14
12	2	0	5	2	2	16	5	6(4)	8	8
15	2	1	8	4	8	33	12	8	14(4)	14
17	2	0	8	1	5	23	14	8	14	11(3)

7. MATHIEU GROUP M_{24} OF ORDER $244823040 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$

In contrast to M_{11}, M_{22} and M_{23}, the Mathieu group M_{24} is the automorphism group of many polytopes, 946 of rank 3, 310 of rank 4, and four of rank 5. The four of rank 5 come as two dual pairs, represented by Schläfli type $\{3,3,10,4\}$ and $\{4,3,10,4\}$ and their duals. As for the rank 5 polytopes of the Higman-Sims group, the vertex figures of these are isomorphic, however, in this case the group of these vertex figures is centreless. The latter group is in fact $M_{22}:2$, which has three polytopes of type $\{3,10,4\}$. The one in question here is the one for which the vertex figure $\{10,4\}$ has group of order 320, with centre $\langle\omega\rangle$ of order 2. (This $\{10,4\}$ is in fact $\{10,4\}*320b$, a quotient of $\{10,4\}_5$.)

The group $\langle s_0', s_1', s_2', s_3', s_4'\rangle$ of the $\{4,3,10,4\}$ may be constructed from the group $\langle s_0, s_1, s_2, s_3, s_4\rangle$ of the $\{3,3,10,4\}$ by allowing $s_i' = s_i$ for $i \neq 0$, and letting $s_0' = s_0\omega$, where ω is the centre of the group of the coface of type $\{10,4\}$

The rank 3 polytopes are summarised in Table 8. Interestingly, of the 310 rank 4 polytopes, not one is self-dual.

Table 8. Rank 3 polytopes for the Mathieu group M_{24}.

	4	5	6	8	10	11	12
4	0	0	0	2	4	14	14
5	0	0	2	0	2	6	10
6	0	2	12(0)	12	16	33	53
8	2	0	12	4(2)	12	10	30
10	4	2	16	12	12(4)	28	54
11	14	6	33	10	28	36(14)	73
12	14	10	53	30	54	73	132(14)

8. McLaughlin group *McL* of order $898128000 = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$

There are no polytopes having the McLaughlin group as their automorphism group.

9. Held group *He* of order $4030387200 = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$

The Held group is the automorphism group of many polytopes, no less than 2292 of rank 3, and 145 of rank 4. The rank 3 polytopes are summarised in Table 9. Amongst the rank 4 polytopes, seven are self-dual, of type $\{4,6,4\}$, $\{4,7,4\}$, $\{5,6,5\}$, $\{6,3,6\}$, $\{6,6,6\}$ (two examples), and $\{6,7,6\}$. The $\{6,3,6\}$ is a quotient of the infinite universal $\{\{6,3\}_{(6,0)}, \{3,6\}_{(6,0)}\}$.

Four of the remaining rank 4 polytopes (two dual pairs) have simple groups as the groups of both the facets and their vertex figures. They are two of type $\{3,15,4\}$ and their duals. The facets of type $\{3,15\}$ have as their group the projective special linear group $L_2(16)$. The vertex figures of type $\{15,4\}$ have group the symplectic group $S_4(4)$.

Table 9. Rank 3 polytopes for the Held group.

	3	4	5	6	7	8	10	12	15	17	21
3	0	0	0	0	0	0	0	3	0	5	8
4	0	0	0	0	0	1	3	14	3	16	28
5	0	0	0	0	0	1	0	6	2	3	6
6	0	0	0	5(1)	2	8	13	53	15	45	61
7	0	0	0	2	0	3	3	5	4	8	7
8	0	1	1	8	3	4(2)	8	21	11	28	27
10	0	3	0	13	3	8	7(3)	47	9	27	39
12	3	14	6	53	5	21	47	121(21)	35	94	118
15	0	3	2	15	4	11	9	35	13(3)	36	42
17	5	16	3	45	8	28	27	94	36	82(22)	104
21	8	28	6	61	7	27	39	118	42	104	116(32)

10. Summary and Conclusions

A web page providing more information on all these polytopes is accessible via [7]. The reader will find there tables giving more detailed information about the polytopes discovered, as well as gzipped GAP [4] files containing, for each polytope, a generating set of its automorphism group as a permutation group.

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