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A GENERALIZATION OF THE q-PFAFF-SAALSCHÜTZ FORMULA

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Abstract. We use the Andrews-Askey integral, the Leibniz rule for the q-difference operator and the q-Chu-Vandermonde formula to give a generalization of the q-Pfaff-Saalschütz formula.

1. Introduction

The following is the well-known q-Pfaff-Saalschütz formula [1, 4]:

$$ {}_3\phi_2\left(\begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{1-n} \end{matrix}; q, q\right) = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}. \tag{1.1} $$

There are many proofs of the q-Pfaff-Saalschütz formula. For example, Chen and Liu used the q-differential operators to give a proof of it [3]. In this paper, we use the Andrews-Askey integral, the Leibniz rule for q-difference operator and the q-Chu-Vandermonde formula to obtain a generalization of the q-Pfaff-Saalschütz formula. The main result is the following theorem:

Theorem 1.1. *Let $0 < |q| < 1$. Then for any nonnegative integers m and n, we have*

$$ \begin{aligned} \sum_{k=0}^{n}\sum_{i=0}^{m} & \frac{(q^{-n};q)_k (q^n;q)_i (q^{-k};q)_{m-i}}{(q;q)_k (c;q)_k (q;q)_{m-i} (q;q)_i}\, q^{k(m+1)} \left(\frac{q^{1-n-k}}{c}\right)^i \\ & \times \int_s^t \frac{\left(\frac{q\omega}{s}, \frac{q\omega}{t}, \frac{a\omega q^{i+1}}{c}, awq^m; q\right)_\infty w^m}{\left(\frac{a\omega q^{1-n}}{c}, a\omega q^{k+i}, u\omega, v\omega; q\right)_\infty}\, d_q\omega \\ & = t(1-q)(-c)^n q^{\binom{n}{2}} \frac{(q, tq/s, s/t, uvst; q)_\infty}{(c;q)_n (us, ut, vs, vt; q)_\infty}\, \delta_{m,0}, \end{aligned} \tag{1.2} $$

provided there are no zero factors in the denominator of the integrals, where

$$ \delta_{m,n} = \begin{cases} 1 & \text{if } m = n, \\ 0, & \text{if } m \neq n, \end{cases} $$

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is the Kronecker delta function.

2. Notation and known results

Before the proof of the theorem, we recall some definitions, notations and known results which will be used in the proof. Throughout this paper, it is supposed that $0<|q|<1$. The q-shifted factorials are defined as

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k). \tag{2.1}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1,a_2,...,a_m;q)_n=(a_1;q)_n(a_2;q)_n...(a_m;q)_n, \tag{2.2}$$

where n is an integer or ∞. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

In 1846 Heine introduced the ${}_{r+1}\phi_r$ basic hypergeometric series, which is defined by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,...,a_{r+1} \\ b_1,b_2,...,b_r\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_{r+1};q)_n x^n}{(q,b_1,b_2,...,b_r;q)_n}. \tag{2.3}$$

The q-difference operator is defined by

$$D_q\{f(a)\}=\frac{1}{a}[f(a)-f(aq)].$$

The following property of D_q is straightforward:

$$D_q^n\left\{\frac{(at;q)_\infty}{(as;q)_\infty}\right\}=s^n\left(\frac{t}{s};q\right)_n\frac{(atq^n;q)_\infty}{(as;q)_\infty}. \tag{2.4}$$

We also have the following Leibniz rule for D_q [6]:

$$D_q^n\{f(a)g(a)\}=\sum_{k=0}^{n}q^{k(k-n)}\begin{bmatrix} n \\ k \end{bmatrix}D_q^k\{f(a)\}D_q^{n-k}\{g(q^ka)\}. \tag{2.5}$$

F. H. Jackson defined the q-integral by [5]

$$\int_0^d f(t)\,d_qt=d(1-q)\sum_{n=0}^{\infty}f(dq^n)q^n,$$

and

$$\int_c^d f(t)\,d_qt=\int_0^d f(t)\,d_qt-\int_0^c f(t)\,d_qt.$$

The following is the Andrews-Askey integral [2] which can be derived from Ramanujan's ${}_1\psi_1$:

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}d_qt=\frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty}, \tag{2.6}$$

provided there are no zero factors in the denominator of the integrals.

In the context of this paper, convergence of q-series is no issue at all because they are the terminating q-series.

3. Proof of the theorem

Using the Andrews-Askey integral and the Leibniz rule for the q-difference operator, the generalization of the q-Pfaff-Saalschütz formula can be easily derived from the q-Chu-Vandermonde convolution formula.

Proof. By the following q-Chu-Vandermonde convolution formula

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q\right) = \frac{a^n (c/a;q)_n}{(c;q)_n}. \tag{3.1} $$

Using the following relation

$$ a^n \left(\frac{c}{a};q\right)_n = (-c)^n q^{\binom{n}{2}} \frac{(aq^{1-n}/c;q)_\infty}{(aq/c;q)_\infty}, \tag{3.2} $$

(3.1) can be written as

$$ \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q,c;q)_k} \cdot \frac{(aq/c,a;q)_\infty}{(aq^{1-n}/c,aq^k;q)_\infty} = (-c)^n \frac{q^{\binom{n}{2}}}{(c;q)_n}. \tag{3.3} $$

Let $a = a\omega$ in (3.3); multiplying the equation (3.3) by

$$ \frac{(q\omega/s,q\omega/t;q)_\infty}{(u\omega,v\omega;q)_\infty}, $$

and taking the q-integral on both sides of (3.3) with respect to variable ω, we get

$$ \begin{aligned} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q,c;q)_k} \cdot \int_s^t \frac{(a\omega q/c,a\omega,q\omega/s,q\omega/t;q)_\infty}{(a\omega q^{1-n}/c,a\omega q^k,u\omega,v\omega;q)_\infty} d_q\omega \\ = (-c)^n \frac{q^{\binom{n}{2}}}{(c;q)_n} \cdot \int_s^t \frac{(q\omega/s,q\omega/t;q)_\infty}{(u\omega,v\omega;q)_\infty} d_q\omega. \end{aligned} \tag{3.4} $$

Using the Andrews-Askey integral (2.6), we have

$$ \int_s^t \frac{(q\omega/s,q\omega/t;q)_\infty}{(u\omega,v\omega;q)_\infty} d_q\omega = \frac{t(1-q)(q,tq/s,s/t,uvst;q)_\infty}{(us,ut,vs,vt;q)_\infty}. \tag{3.5} $$

Substituting (3.5) into (3.4), we get

$$ \begin{aligned} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q,c;q)_k} \cdot \int_s^t \frac{(q\omega/s,q\omega/t;q)_\infty}{(u\omega,v\omega;q)_\infty} \cdot \frac{(a\omega q/c,a\omega;q)_\infty}{(a\omega q^{1-n}/c,a\omega q^k;q)_\infty} d_q\omega \\ = t(1-q)(-c)^n q^{\binom{n}{2}} \frac{(q,tq/s,s/t,uvst;q)_\infty}{(c;q)_n (us,ut,vs,vt;q)_\infty}. \end{aligned} \tag{3.6} $$

Applying D_q^m on both sides of the above identity with respect to variable a, we get

$$D_q^m\left\{\sum_{k=0}^{n}\frac{(q^{-n};q)_kq^k}{(q,c;q)_k}\cdot\int_s^t\frac{(q\omega/s,q\omega/t;q)_\infty}{(u\omega,v\omega;q)_\infty}\cdot\frac{(a\omega q/c,a\omega;q)_\infty}{(a\omega q^{1-n}/c,a\omega q^k;q)_\infty}d_q\omega\right\} = D_q^m\left\{t(1-q)(-c)^nq^{\binom{n}{2}}\frac{(q,tq/s,s/t,uvst;q)_\infty}{(c;q)_n(us,ut,vs,vt;q)_\infty}\right\}. \tag{3.7}$$

Using (3.6) and noticing $D_q^m\{f(a)\}$ is a linear combination of $f(aq^j)-f(aq^{j+1})$, we get

$$\begin{aligned}&D_q^m\left\{\sum_{k=0}^{n}\frac{(q^{-n};q)_kq^k}{(q,c;q)_k}\cdot\int_s^t\frac{(q\omega/s,q\omega/t;q)_\infty}{(u\omega,v\omega;q)_\infty}\cdot\frac{(a\omega q/c,a\omega;q)_\infty}{(a\omega q^{1-n}/c,a\omega q^k;q)_\infty}d_q\omega\right\}\\ &=\sum_{k=0}^{n}\frac{(q^{-n};q)_kq^k}{(q,c;q)_k}\cdot\int_s^t\frac{(q\omega/s,q\omega/t;q)_\infty}{(u\omega,v\omega;q)_\infty}\cdot D_q^m\left\{\frac{(a\omega q/c,a\omega;q)_\infty}{(a\omega q^{1-n}/c,a\omega q^k;q)_\infty}\right\}d_q\omega.\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we get

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_kq^k}{(q,c;q)_k}\cdot\int_s^t\frac{(q\omega/s,q\omega/t;q)_\infty}{(u\omega,v\omega;q)_\infty}\cdot D_q^m\left\{\frac{(a\omega q/c,a\omega;q)_\infty}{(a\omega q^{1-n}/c,a\omega q^k;q)_\infty}\right\}d_q\omega = D_q^m\left\{t(1-q)(-c)^nq^{\binom{n}{2}}\frac{(q,tq/s,s/t,uvst;q)_\infty}{(c;q)_n(us,ut,vs,vt;q)_\infty}\right\}. \tag{3.9}$$

The right hand side of (3.9) equals

$$D_q^m\left\{t(1-q)(-c)^nq^{\binom{n}{2}}\frac{(q,tq/s,s/t,uvst;q)_\infty}{(c;q)_n(us,ut,vs,vt;q)_\infty}\right\} = t(1-q)(-c)^nq^{\binom{n}{2}}\frac{(q,tq/s,s/t,uvst;q)_\infty}{(c;q)_n(us,ut,vs,vt;q)_\infty}\delta_{m,\,0}. \tag{3.10}$$

Now we begin to calculate the left hand side of (3.9). By (2.5), We have

$$D_q^m\left\{\frac{(a\omega q/c,a\omega;q)_\infty}{(a\omega q^{1-n}/c,a\omega q^k;q)_\infty}\right\} = \sum_{i=0}^{m}q^{i(i-m)}\begin{bmatrix}m\\ i\end{bmatrix}D_q^i\left\{\frac{(a\omega q/c;q)_\infty}{(a\omega q^{1-n}/c;q)_\infty}\right\}D_q^{m-i}\left\{\frac{(aq^i\omega;q)_\infty}{(a\omega q^{k+i};q)_\infty}\right\}. \tag{3.11}$$

Using (2.4) to get

$$D_q^i\left\{\frac{(a\omega q/c;q)_\infty}{(a\omega q^{1-n}/c;q)_\infty}\right\}=\frac{(\omega q^{1-n}/c)^i(q^n;q)_i(awq^{i+1}/c;q)_\infty}{(a\omega q^{1-n}/c;q)_\infty} \tag{3.12}$$

and

$$D_q^{m-i}\left\{\frac{(aq^i\omega;q)_\infty}{(a\omega q^{k+i};q)_\infty}\right\}=\frac{(\omega q^{k+i})^{m-i}(q^{-k};q)_{m-i}(awq^m;q)_\infty}{(a\omega q^{k+i};q)_\infty}. \tag{3.13}$$

Substituting (3.10), (3.11), (3.12) and (3.13) into (3.9), we get the left hand side of (3.9)

(3.14)

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(q^{-n};q)_kq^k}{(q,c;q)_k}\cdot\int_s^t\frac{(q\omega/s,q\omega/t;q)_\infty}{(u\omega,v\omega;q)_\infty}\cdot D_q^m\left\{\frac{(a\omega q/c,a\omega;q)_\infty}{(a\omega q^{1-n}/c,a\omega q^k;q)_\infty}\right\}d_q\omega\\&=\sum_{k=0}^{n}\sum_{i=0}^{m}\frac{(q^{-n};q)_k(q^n;q)_i(q^{-k};q)_{m-i}(q;q)_m}{(q\,;q)_k\ (c\,;q)_k\ (q\,;q)_{m-i}\ (q\,;q)_i}\ q^{k(m+1)}(q^{1-n-k}/c)^i\\&\qquad\times\int_s^t\frac{(q\omega/s,q\omega/t,a\omega q^{i+1}/c,awq^m;q)_\infty w^m}{(a\omega q^{1-n}/c,a\omega q^{k+i},u\omega,v\omega;q)_\infty}\ d_q\omega.\end{aligned}$$

Combining (3.9),(3.10) and (3.14), we get

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{i=0}^{m}\frac{(q^{-n};q)_k(q^n;q)_i(q^{-k};q)_{m-i}(q;q)_m}{(q\,;q)_k\ (c\,;q)_k\ (q\,;q)_{m-i}\ (q\,;q)_i}\ q^{k(m+1)}(q^{1-n-k}/c)^i\\&\qquad\times\int_s^t\frac{(q\omega/s,q\omega/t,a\omega q^{i+1}/c,awq^m;q)_\infty w^m}{(a\omega q^{1-n}/c,a\omega q^{k+i},u\omega,v\omega;q)_\infty}\ d_q\omega\\&\qquad\qquad=t(1-q)(-c)^nq^{\binom{n}{2}}\frac{(q,tq/s,s/t,uvst;q)_\infty}{(c;q)_n(us,ut,vs,vt;q)_\infty}\ \delta_{m,\,0},\end{aligned}\tag{3.15}$$

which is equivalent to (1.2). □

4. A corollary of the theorem

In this section, we point out that the q-Pfaff-Saalschütz formula is a special case of (1.2).

Corollary 4.1. *(The q-Pfaff-Saalschütz formula) We have*

$${}_3\phi_2\left(\begin{matrix}a,b,q^{-n}\\c,abc^{-1}q^{1-n}\end{matrix};q,q\right)=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}. \tag{4.1}$$

Proof. Let $u=aq/c,v=a$ and $m=0$ in (1.2) to get

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(q^{-n};q)_kq^k}{(q,c;q)_k}\cdot\int_s^t\frac{(q\omega/s,q\omega/t;q)_\infty}{(a\omega q^k,a\omega q^{1-n}/c;q)_\infty}d_q\omega\\&\qquad=t(1-q)(-c)^nq^{\binom{n}{2}}\frac{(q,tq/s,s/t,a^2stq/c;q)_\infty}{(c;q)_n(asq/c,atq/c,as,at;q)_\infty}.\end{aligned}\tag{4.2}$$

Using the Andrews-Askey integral (2.6), we have

(4.3)

$$\int_s^t\frac{(q\omega/s,q\omega/t;q)_\infty}{(a\omega q^k,a\omega q^{1-n}/c;q)_\infty}d_q\omega=\frac{t(1-q)(q,tq/s,s/t,a^2stq^{k+1-n}/c;q)_\infty}{(aq^ks,aq^kt,asq^{1-n}/c,atq^{1-n}/c;q)_\infty}.$$

Substituting (4.3) into (4.2), we get

$$(4.4)\quad \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q,c;q)_k} \cdot \frac{t(1-q)(q,tq/s,s/t,a^2stq^{k+1-n}/c;q)_\infty}{(aq^k s, aq^k t, asq^{1-n}/c, atq^{1-n}/c;q)_\infty} = t(1-q)(-c)^n q^{\binom{n}{2}} \frac{(q,tq/s,s/t,a^2stq/c;q)_\infty}{(c;q)_n (asq/c, atq/c, as, at;q)_\infty}.$$

After replacing (as, at) by (a, b) and employing the following formula

$$(a;q)_n = (a^{-1};q^{-1})_n (-a)^n q^{\binom{n}{2}},$$

we get the q-Pfaff-Saalschütz formula (4.1). □

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