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EXPLICIT UPPER BOUNDS FOR $\prod_{p\leq p_{\omega(n)}} p/(p-1)$

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Abstract. Let $\omega(n)$ be the number of distinct prime divisors of n, $\phi(n)$ the Euler totient function, $\sigma(n)$ the sum of divisors of n, p_n the n^{th} prime and γ the Euler constant. We consider the arithmetic function

$$f(n) = \prod_{\substack{p\leq p_{\omega(n)}\\ p \text{ prime}}} \frac{p}{p-1}$$

and show that

$$\overline{\lim} \frac{f(n)}{e^{\gamma}\log\log n} = 1.$$

Next we describe an algorithm that, for a given $0 < \epsilon < 1$, determines all the exceptions to the inequality

$$f(n) < e^{\gamma}(1+\epsilon)\log\log n.$$

Finally, by employing this algorithm, we establish some explicit upper bounds for $n/\phi(n)$ and $\sigma(n)/n$. More specifically, we prove that

$$\frac{\sigma(n)}{n\log\log n} \leq \frac{\sigma(180)}{180\log\log 180} = (1.0338\ldots)e^{\gamma} \text{ for } n \geq 121.$$

1. Introduction

Let $\sigma(n)$ denote the sum of divisors function and $\phi(n)$ the Euler totient function, so that $\sigma(n)\phi(n) < n^2$. Nicolas [4] proved that $n/\phi(n) > e^{\gamma}\log\log n$ infinitely often, where γ is the Euler constant. Also for the smaller quantity $\sigma(n)/n$, Robin showed that $\sigma(n)/n > e^{\gamma}\log\log n$ infinitely often provided the Riemann Hypothesis is false. More precisely, let $g(n) = \sigma(n)/n\log\log n$. Then in [5], Robin proved the following theorem.

Theorem 1.1 (Robin). *The Riemann Hypothesis is true if and only if*

$$g(n) < e^{\gamma} \text{ for } n \geq 5041.$$

As a consequence of this theorem, one can show that under the assumption of the Riemann Hypothesis, the only values of n that fail $g(n) < e^{\gamma}$ are

$$n \in \{2,3,4,5,6,8,9,10,12,16,18,20,24,30,36,48,60,72,84, 120,180,240,360,720,840,2520,5040\}$$

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(see [5], p. 204). Table 1 shows the values of $g(n)$ in decreasing order for the above exceptions (other than $n = 2$) to the inequality $g(n) < e^{\gamma}$.

n	$g(n)$	n	$g(n)$	n	$g(n)$
3	14.177...	60	1.986...	9	1.834...
4	5.357...	36	1.980...	360	1.833...
6	3.429...	30	1.960...	240	1.822...
12	2.563...	120	1.915...	2520	1.804...
8	2.561...	20	1.913...	840	1.797...
5	2.521...	48	1.908...	84	1.791...
24	2.162...	16	1.899...	5040	1.790...
10	2.158...	72	1.863...	720	1.782...
18	2.041...	180	1.841...		

TABLE 1. Values of $g(n)$.

The following assertion is a direct corollary of Robin's theorem together with the values recorded in Table 1.

Corollary 1.2. *Under the assumption of the Riemann Hypothesis, we have the following inequalities:*

$$g(n) \leq g(3) = (7.959914266\ldots)e^{\gamma} \quad \text{for } n \geq 3, \tag{1.1}$$

$$g(n) \leq g(4) = (3.008117079\ldots)e^{\gamma} \quad \text{for } n \geq 4, \tag{1.2}$$

$$g(n) \leq g(6) = (1.925450381\ldots)e^{\gamma} \quad \text{for } n \geq 5, \tag{1.3}$$

$$g(n) \leq g(12) = (1.439267874\ldots)e^{\gamma} \quad \text{for } n \geq 7, \tag{1.4}$$

$$g(n) \leq g(24) = (1.213946496\ldots)e^{\gamma} \quad \text{for } n \geq 13, \tag{1.5}$$

$$g(n) \leq g(60) = (1.115266133\ldots)e^{\gamma} \quad \text{for } n \geq 25, \tag{1.6}$$

$$g(n) \leq g(120) = (1.075588326\ldots)e^{\gamma} \quad \text{for } n \geq 61, \tag{1.7}$$

$$g(n) \leq g(180) = (1.033867784\ldots)e^{\gamma} \quad \text{for } n \geq 121, \tag{1.8}$$

$$g(n) \leq g(360) = (1.029419689\ldots)e^{\gamma} \quad \text{for } n \geq 181, \tag{1.9}$$

$$g(n) \leq g(2520) = (1.013215898\ldots)e^{\gamma} \quad \text{for } n \geq 361, \tag{1.10}$$

$$g(n) \leq g(5040) = (1.005558981\ldots)e^{\gamma} \quad \text{for } n \geq 2521. \tag{1.11}$$

One of our goals in this paper is to prove some of the above inequalities unconditionally. Note that establishing an inequality in one line establishes all the inequalities in the previous lines. The inequality (1.4) is proved by Robin ([5], Proposition 2). This inequality gives an improvement of a previous result of Ivić [3]. Here we will prove the following.

Theorem 1.3. $g(n) \leq g(180) = (1.0338\ldots)e^{\gamma}$ *for* $n \geq 121$. *More precisely,* $g(n) < 1.03e^{\gamma}$ *for* $n \geq 121$ *except for* $n = 180$.

The methodology of the proof is as follows. Let

$$f(n) = \prod_{\substack{p\le p_{\omega(n)}\\ p \text{ prime}}} \frac{p}{p-1},$$

where $\omega(n)$ is the number of distinct prime divisors of n and p_n is the n^{th} prime. We develop an algorithm that generates all the exceptions to the inequality

$$f(n) < 1.03e^{\gamma}\log\log n.$$

Since $\sigma(n)/n < f(n)$, it is clear that the exceptions to

$$g(n) \le g(180) = (1.0338\dots)e^{\gamma}$$

are among the exceptions to

$$f(n) < 1.03e^{\gamma}\log\log n.$$

So, numerically checking the exceptions generated by the algorithm against the inequality

$$g(n) < 1.03e^{\gamma}$$

will establish the result.

The structure of the paper is as follows. In Section 2, we study some properties of the arithmetic function $f(n)$. We describe the algorithm in Section 3. In the last section, we employ our algorithm to establish an explicit upped bound for $n/\phi(n)$ and prove Theorem 1.3.

2. The Arithmetic Function $f(n)$

Let

$$f(n) = \prod_{\substack{p\le p_{\omega(n)}\\ p \text{ prime}}} \frac{p}{p-1}.$$

We have

$$\frac{\sigma(n)}{n} < \frac{n}{\phi(n)} \le f(n).$$

The right-hand side inequality is trivial and the left-hand side one is Theorem 329 of [2].

For a given n, let $p_1,\dots,p_{\omega(n)-l}$ be the primes less than or equal to $\log n$ and $p_{\omega(n)-l+1},\dots,p_{\omega(n)}$ be those that exceed $\log n$. One can show that

$$l < \frac{\log n}{\log\log n}$$

and

$$f(n) < \left(1-\frac{1}{\log n}\right)^{-\log n/\log\log n} \prod_{p\le \log n}\frac{p}{p-1}. \tag{2.1}$$

Proposition 2.1.
$$\overline{\lim} \frac{f(n)}{e^\gamma \log\log n} = 1.$$

Proof. First of all, recall the Mertens theorem:
$$\prod_{p\leq x} \frac{p}{p-1} \sim e^\gamma \log x$$
as $x \to \infty$ ([2], Theorem 429). Now, an application of the Mertens theorem to (2.1) implies
$$\overline{\lim} \frac{f(n)}{e^\gamma \log\log n} \leq 1.$$
Let $n_x = \prod_{p\leq x} p$. By the Prime Number Theorem, we know that $\log\log n_x \sim \log x$ as $x \to \infty$. So, by the Mertens theorem,
$$\lim_{x\to\infty} \frac{f(n_x)}{e^\gamma \log\log n_x} = 1.$$
This completes the proof. □

Corollary 2.2. *For any $\epsilon > 0$, there exists a number N_ϵ such that*
$$f(n) < e^\gamma(1+\epsilon)\log\log n \tag{2.2}$$
for all $n > N_\epsilon$.

Our next goal is to establish explicit versions of (2.2). More precisely, for a given $\epsilon > 0$, we want to find an algorithm that finds the smallest value for N_ϵ. In the next section, we describe an algorithm that, for a given $\epsilon > 0$, generates all the exceptions to (2.2).

3. Explicit Upper Bounds for $f(n)$

For $\epsilon > 0$, let
$$M_\epsilon = \exp\left(\exp\left(\sqrt{\frac{2.50637}{\epsilon e^\gamma}}\right)\right).$$

Lemma 3.1. *Equation (2.2) holds for $n > M_\epsilon$ with $\omega(n) \geq 3$.*

Proof. Let $Q_{\omega(n)} = p_1 \cdots p_{\omega(n)}$. Then by Theorem 15 of [6], we have
$$f(n) = \prod_{p\,|\,Q_{\omega(n)}} \frac{p}{p-1} = \frac{Q_{\omega(n)}}{\phi(Q_{\omega(n)})} < e^\gamma \log\log Q_{\omega(n)} + \frac{2.50637}{\log\log Q_{\omega(n)}}$$
for n with $\omega(n) > 1$. Note that $Q_{\omega(n)} \leq n$ and the function
$$g(t) = e^\gamma t + \frac{2.50637}{t}$$
is increasing for $t \geq 1.2$. So for any n with $\omega(n) \geq 3$, we have
$$f(n) < e^\gamma \log\log n + \frac{2.50637}{\log\log n}.$$
From here, it is clear that (2.2) holds for any $n > M_\epsilon$ with $\omega(n) \geq 3$. □

For an integer $\beta \ge 1$, let

$$n_\beta = \exp\left(\exp\left(\frac{1}{(1+\epsilon)e^\gamma}\prod_{p\le p_\beta}\frac{p}{p-1}\right)\right).$$

Note that n_β depends on ϵ. For simplicity, we use n_β instead of $n_{\beta,\epsilon}$.

Lemma 3.2. *Equation* (2.2) *holds for all* $n > n_\beta$ *with* $\omega(n) \le \beta$.

Proof. Let $n > n_\beta$ with $\omega(n) \le \beta$. Then,

$$f(n) \le \prod_{p\le p_\beta}\frac{p}{p-1} < e^\gamma(1+\epsilon)\log\log n.$$

□

Since $n_2 < M_\epsilon$ for $0<\epsilon<1$, the following is a direct corollary of Lemmas 3.1 and 3.2.

Corollary 3.3. *Let* $0<\epsilon<1$. *Then for* $n > M_\epsilon$, (2.2) *holds.*

Next, we give a description of all exceptions to the inequality

$$f(n) < e^\gamma(1+\epsilon)\log\log n$$

with $\omega(n) \le \beta$.

Lemma 3.4. *Let* $\epsilon > 0$ *be real and* $\beta \ge 2$ *be an integer.*

(1) *If* $\prod_{p\le p_\beta} p > n_\beta$, *then* (2.2) *holds for all* $n_{\beta-1} < n \le n_\beta$.
(2) *If* $\prod_{p\le p_\beta} \le n_\beta$, *then integers with* β *distinct prime divisors not exceeding* n_β *do not satisfy* (2.2).

Proof. (1) If $\prod_{p\le p_\beta} p > n_\beta$, then for any $n \le n_\beta$ we have $\omega(n) \le \beta - 1$. So, by Lemma 3.2, inequality (2.2) holds for $n_{\beta-1} < n \le n_\beta$.

(2) If n is an integer with exactly β prime factors and $n \le n_\beta$, then

$$f(n) = \prod_{p\le p_\beta}\frac{p}{p-1} \ge e^\gamma(1+\epsilon)\log\log n.$$

□

We are now ready to describe our algorithm.

Algorithm 3.5 Finds exceptions to $f(n) < e^\gamma(1+\epsilon)\log\log n$.

Input: $0<\epsilon<1$.
Output: All exceptions to the inequality.

1: Calculate M_ϵ.
2: Find the largest β such that

$$\prod_{p\le p_\beta} p \le M_\epsilon.$$

3: **while** $\prod_{p\le p_\beta} p > n_\beta$ **do**

4: $\beta \leftarrow \beta - 1$.
5: **end while**
6: Calculate n_α for $1 \leq \alpha \leq \beta$.
7: For any $1 \leq \alpha \leq \beta$, find all integers with α distinct prime divisors not exceeding n_α and write them in a file.

The correctness of this algorithm is a direct consequence of the previous lemmas. From now on, we call an exception to the inequality $f(n) < e^\gamma(1+\epsilon)\log\log n$ simply an *exception.*

Example 3.6. *For $\epsilon = 0.07$ we have*

$$[M_{0.07}] = 2886575284525970951227105717034435368 40.$$

Since 26 is the greatest number such that $\prod_{p \leq p_\beta} p \leq M_{0.07}$, we have 26 as the initial value of β. After the while-loop executes, the algorithm will give 18 as the final value of β. Then for any α $(1 \leq \alpha \leq \beta)$, Step 7 of the algorithm constructs all the exceptions. Table 2 records the number of exceptions for each α.

For example, there are 212 exceptions with $\omega(n) = 18$. In total, there are $3,281,014$ *exceptions to the inequality $f(n) \leq 1.07e^\gamma \log\log n$.*

$\alpha = \omega(n)$	# of Exceptions	$\alpha = \omega(n)$	# of Exceptions
1	11	10	542533
2	69	11	740427
3	373	12	564065
4	2319	13	329802
5	7418	14	210907
6	27134	15	106791
7	66268	16	27963
8	197450	17	3043
9	454229	18	212

Table 2. Number of exceptions for each α.

4. Applications

We recall that

$$\frac{\sigma(n)}{n} < \frac{n}{\phi(n)} \leq f(n).$$

Moreover,

$$\overline{\lim}\frac{\sigma(n)}{ne^\gamma \log\log n} = \overline{\lim}\frac{n}{\phi(n)e^\gamma \log\log n} = \overline{\lim}\frac{f(n)}{e^\gamma \log\log n} = 1$$

(see [2], p. 353 and Proposition 2.1). We can use the above inequality, together with our algorithm, to establish explicit upper bounds for $\sigma(n)/n$ and $n/\phi(n)$. For example, by numerically checking the exceptions generated by the algorithm for $\epsilon = 0.07$ against the inequality $n/\phi(n) < 1.07e^{\gamma} \log\log n$, we find that there are 6,569 exceptions to $n/\phi(n) < 1.07e^{\gamma} \log\log n$, the largest being 23457676271881394196654. This gives us the following proposition.

Proposition 4.1.

$$\frac{n}{\phi(n)} < 1.07e^{\gamma} \log\log n \text{ for } n > 23457676271881394196654 0.$$

Using the same method, we deduce that there are only 14 exceptions to the inequality $\sigma(n)/n < 1.07e^{\gamma} \log\log n$. The largest of these is 120. Hence,

$$\frac{\sigma(n)}{n} < 1.07e^{\gamma} \log\log n \text{ for } n \ge 121.$$

As expected, the number of exceptions increases dramatically as ϵ gets smaller. For example, for $\epsilon = 0.06$ there are 32,707,736 exceptions and for $\epsilon = 0.05$ there are 798,101,126 exceptions. If we are only interested in upper bounds for $\sigma(n)$, we can reduce the number of possible exceptions to the inequality $\sigma(n) < e^{\gamma}(1+\epsilon) \log\log n$ significantly by the following two observations.

First of all, by a result of Robin ([5], Theorem 2), for $n \ge 3$, we have

$$\frac{\sigma(n)}{n} \le e^{\gamma} \log\log n + \frac{0.6483}{\log\log n}.$$

So, by an argument similar to Lemma 3.1,

$$\frac{\sigma(n)}{n} < e^{\gamma}(1+\epsilon) \log\log n \tag{4.1}$$

for

$$n > \overline{M}_{\epsilon} = \exp\left(\exp\left(\sqrt{\frac{0.6483}{\epsilon e^{\gamma}}}\right)\right)$$

with $\omega(n) \ge 2$. Note that $\overline{M}_{\epsilon}$ is much smaller that M_{ϵ}. Thus we need only check (4.1) for the exceptions not exceeding $\overline{M}_{\epsilon}$.

Secondly, by a recent result of Choie, Lichiardopol, Moree and Solé [1], we know that if $n \ge 5041$ does not satisfy Robin's inequality $\sigma(n) > e^{\gamma} n \log\log n$, then n is even, is neither squarefree nor squarefull, and is divisible by a fifth power greater than 1. In other words, all $n \ge 5041$ that are odd, squarefree, squarefull, or are not divisible by a fifth power greater than 1 satisfy (4.1).

Let S_1 be the set of natural numbers not exceeding $\overline{M}_{\epsilon}$. Let S_2 be those numbers in S_1 that are even, neither squarefree nor squarefull, and are divisible by a fifth power greater than 1. Table 3 gives the number of exceptions, the number of exceptions in S_1 and the number of exceptions in S_2 for different values of ϵ.

Notice that in order to prove Theorem 1.3, we need to check (4.1) in the case $\epsilon = 0.03$ for about 55,000,000 values of n.

ϵ	# of exceptions	# of exceptions in S_1	# of exceptions in S_2
0.07	3,281,014	10,190	135
0.06	32,707,734	63,076	850
0.05	798,101,116	418,627	5,672
0.04		23,472,726	323,069
0.03		4,420,980,851	55,258,878
0.029		15,910,840,055	183,084,959

TABLE 3. Number of exceptions total, in S_1, and in S_2.

Proof of Theorem 1.3. By running our algorithm for $\epsilon = 0.03$, we generate all the exceptions to the inequality $f(n) < 1.03e^{\gamma} \log\log n$ in S_2. By checking these exceptions against the inequality $g(n) < 1.03e^{\gamma}$, we find that 180 is the largest n that does not satisfy $g(n) < 1.03e^{\gamma}$. $\square$

We end with a note on the computations. The algorithm is implemented in C++. The checking of exceptions against the inequality $g(n) < (1+\epsilon)e^{\gamma}$ is done by Maple 10. The running time for checking in the case $\epsilon = 0.03$ is 40 hours with a 3 GHz Intel Pentium 3 having 1 GB of memory.

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