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PARTITIONS WITH PARTS OCCURRING AT MOST THRICE

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In memoriam Johannes van Lint.

Abstract. We study partitions of n into parts that occur at most thrice, with weights whose definition is motivated by an identity of Jacobi. A combinatorial bijection between odd and even partitions of maximum weight is extended to a bijection of "potholes" (partitions supplied with extra structure) which is used to show that, when n is not triangular, the numbers of odd and even partitions of any weight are equal. The situation for triangular numbers is also analyzed, and this provides a new proof of Jacobi's identity. Finally, the numbers of potholes are related to a Jacobi theta function, and several other combinatorial connections are noted.

1. Introduction

In [9, p. 37] Marshall Hall made the following comment concerning Jacobi's identity:

$$\prod_{i=1}^{\infty}(1-q^i)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2} \tag{1.1}$$

> *"Although the product on the left can be associated with partitions into parts, no one of which may be used more than three times, no combinatorial proof of this identity is known to the writer."*

In [11] Joichi and Stanton gave a combinatorial proof: they interpreted the left side of (1.1) as the generating function for ordered triples of partitions into distinct parts with each part weighted by -1. They then gave an involution on the set of those ordered triples which reverses signs except for special partitions of triangular numbers, which are fixed.

In this paper we equip a partition whose part sizes occur at most thrice with a **weight**, 3^l, where l is the total number of part sizes which occur just once or twice, and also with a **sign**, $\pm$, according as the total number of

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parts is even or odd. Except in Section 3, "partition" will always mean a partition in which no part size occurs more than thrice.

In Section 2 we pursue the line of thought mentioned in [9, p. 37] and interpret the Jacobi product in (1.1) as the generating function for such partitions. It appears that for each weight, unless n is triangular, the numbers of even (+) and odd (–) partitions are equal. This is proved in Section 10.

In Section 3 we prove that the numbers of these partitions satisfy a very close analog of Euler's pentagonal number theorem.

In Section 4 we observe that, between two consecutive triangular numbers, for say $\frac{1}{2}k(k+1) < n < \frac{1}{2}(k+1)(k+2)$, the numbers of odd and even partitions of weight k always belong to the same sequence (4.1). We verify this by means of a specific bijection between maximum weight partitions of n and similar such partitions of $n+k+1$ which have opposite parity. This construction is generalized in Section 11.

In Section 5 we examine the exceptional case of the triangular numbers, which is the only case where the numbers of odd and even partitions of given weight differ. The differences are observed to satify a recurrence, which is proved in Section 10.

In Sections 6 and 7 we set up the scaffolding for the bijection between "potholes" which are introduced, and illustrated by examples, in Section 8. Potholes are defined as partitions whose part sizes are assigned a variety of "heights". In Section 9 we show how the bijection of potholes induces a combinatorial bijection on partitions of maximum weight.

In Section 10 we prove the recurrence (5.1) observed in Section 5. In Section 11, we generalize the bijection of Section 4. We also give a further interpretation of sequence (4.1), which is related to a Jacobi theta function in Section 12. Finally, in Section 13 we observe many other combinatorial connections with the differences of Section 5.

2. The Jacobi product as a generating function

We recall Marshall Hall's remark and rewrite the left side of (1.1) as

$$\prod_{i=1}^{\infty}(1-3q^i+3q^{2i}-q^{3i}) \tag{2.1}$$

This is the generating function for our partitions, weighted by a factor 3 whenever a part size occurs just once or twice, and with partitions counted positively or negatively according as they are **even** or **odd**, that is, whether the total number of parts is even or odd.

For example, the partitions of 4 in which we are interested, with superscripts denoting multiplicity of part size, are the following:

	4^1	$3^1 1^1$	2^2	$2^1 1^2$
weight	3	9	3	9
parity	odd	even	even	odd

Accordingly, the coefficient of q^4 in (2.1) is $-3+9+3-9=0$. We note that the number of even partitions is equal to the number of odd partitions.

On the other hand, for the triangular number 6 we get

	6^1	$5^1 1^1$	$4^1 2^1$	$4^1 1^2$	3^2	$3^1 2^1 1^1$	$3^1 1^3$	2^3	$2^2 1^2$
weight	3	9	9	9	3	27	3	1	9
parity	odd	even	even	odd	even	odd	even	odd	even

and $-3+9+9-9+3-27+3-1+9=-7$ matches the coefficient $(-1)^k(2k+1)$ of $q^{k(k+1)/2}$ for $k=3$ on the right side of (1.1).

We write $n=\frac{1}{2}k(k+1)+m$ where $0 \le m \le k$. Table 1 lists the numbers of partitions of n with parts occurring at most thrice, with weight 3^l, where $t^+_{n,l}$ counts the number of partitions with even multiplicity while $t^-_{n,l}$ counts the odd partitions. In the above examples, for $n=4$, t^+ counts the even partitions 2^2 of weight 3^1 and $3^1 1^1$ of weight 3^2 and t^- counts the odd partitions 4^1 of weight 3^1 and $2^1 1^2$ of weight 3^2. For $n=6$, t^+ counts the even partitions 3^2 and $3^1 1^3$ of weight 3^1, and $5^1 1^1$, $4^1 2^1$, $2^2 1^2$, each of weight 3^2, while t^- counts the odd partitions 2^3 of weight 3^0, 6^1 of weight 3^1, $4^1 1^2$ of weight 3^2, and $3^1 2^1 1^1$ of weight 3^3. The generating function for these weighted and signed partitions can be written

$$\prod_{i=1}^{\infty}(1-3q^i+3q^{2i}-q^{3i}) = \sum_{n=0}^{\infty}\sum_{l\ge 0} 3^l (t^+_{n,l}-t^-_{n,l})q^n$$

Table 1 suggests the following theorem.

Theorem 2.1. *Unless n is triangular (the* ***bold*** *entries in the second column), the numbers $t^-_{n,l}$ and $t^+_{n,l}$ are equal. In particular, the totals*

$$T^+_n = \sum_{l\ge 0} t^+_{n,l} \quad \text{and} \quad T^-_n = \sum_{l\ge 0} t^-_{n,l}$$

are equal except for triangular n. If $n=\frac{1}{2}k(k+1)$ is triangular, then $T^+_n - T^-_n = (-1)^n$. Furthermore,

$$\sum_{l\ge 0} 3^l (t^+_{n,l}-t^-_{n,l}) = \begin{cases} 0 & \text{if } n \text{ is not triangular,} \\ (-1)^k(2k+1) & \text{if } n=\frac{1}{2}k(k+1). \end{cases}$$

Theorem 2.1 provides another proof of Jacobi's identity (1.1). It will be proved in Section 10.

Table 1 strongly suggests the existence of a natural bijection between even and odd partitions of each weight, when n is not triangular. We have only been able to give a direct construction of such a bijection for the special case of maximal weight.

weight		3^0		3^1		3^2		3^3		3^4		3^5		3^6		3^7		Totals		
k	n	t^+	t^-	t^+	t^-	t^+	t^-	t^+	t^-	t^+	t^-	t^+	t^-	t^+	t^-	t^+	t^-	T_n^+	T_n^-	Sum
0	**0**	1	0															1	0	1
1	**1**	0	0	0	1													0	1	1
	2	0	0	1	1													1	1	2
2	**3**	0	1	0	1	1	0											1	2	3
	4	0	0	1	1	1	1											2	2	4
	5	0	0	1	1	2	2											3	3	6
3	**6**	0	1	2	1	3	1	0	1									5	4	9
	7	0	0	2	2	3	3	1	1									6	6	12
	8	0	0	2	2	4	4	2	2									8	8	16
	9	1	1	2	2	4	4	4	4									11	11	22
4	**10**	0	0	4	1	7	7	3	6	1	0							15	14	29
	11	0	0	3	3	8	8	7	7	1	1							19	19	38
	12	1	1	3	3	9	9	10	10	2	2							25	25	50
	13	0	0	4	4	12	12	12	12	4	4							32	32	64
	14	0	0	5	5	14	14	15	15	7	7							41	41	82
5	**15**	2	1	5	3	13	18	21	23	11	7	0	1					52	53	105
	16	0	0	6	6	19	19	26	26	14	14	1	1					66	66	132
	17	0	0	7	7	22	22	32	32	20	20	2	2					83	83	166
	18	2	2	6	6	24	24	42	42	26	26	4	4					104	104	208
	19	0	0	8	8	30	30	49	49	35	35	7	7					129	129	258
	20	0	0	8	8	36	36	60	60	44	44	12	12					160	160	320
6	**21**	3	2	8	10	33	40	76	70	62	57	14	19	1	0			197	198	395
	22	0	0	11	11	45	45	85	85	75	75	25	25	1	1			242	242	484
	23	0	0	12	12	52	52	100	100	94	94	36	36	2	2			296	296	592
	24	3	3	12	12	55	55	120	120	118	118	49	49	4	4			361	361	722
	25	0	0	14	14	67	67	138	138	145	145	67	67	7	7			438	438	876
	26	0	0	16	16	74	74	161	161	179	179	88	88	12	12			530	530	1060
	27	4	4	16	16	76	76	192	192	220	220	112	112	20	20			640	640	1280
7	**28**	0	0	17	22	94	94	222	207	260	265	146	155	31	25	0	1	770	769	1539

TABLE 1. Numbers $t^{\pm}_{n,l}$ of partitions of n with parts occurring at most thrice, listed by parity and weight. Note the similarity of the first nonzero entries in each 3^l column.

3. A recurrence like Euler's

The sequences T_n^- and T_n^+ of totals of odd and even partitions with no parts occurring more than thrice,

$$0, 1, 1, 2, 2, 3, 4, 6, 8, 11, 14, 19, 25, 32, 41, 53, 66, \ldots$$
$$1, 0, 1, 1, 2, 3, 5, 6, 8, 11, 15, 19, 25, 32, 41, 52, 66, \ldots$$

were not, at the time of writing, in OEIS [17], but their sum,

$$1, 1, 2, 3, 4, 6, 9, 12, 16, 22, 29, 38, 50, 64, 82, 105, \ldots \tag{3.1}$$

is sequence A001935, where we learn that these are also the numbers of partitions of n with no part a multiple of 4, and the numbers of partitions with no even part repeated.

These equalities are easily seen ([10, pp. 68, 241] or [16]) from the generating function, which may be written variously as

$$\begin{aligned}
&(1+q+q^2+q^3)(1+q^2+q^4+q^6)(1+q^3+q^6+q^9)(1+q^4+q^8+q^{12})\cdots \\
&= \frac{1-q^4}{1-q}\cdot\frac{1-q^8}{1-q^2}\cdot\frac{1-q^{12}}{1-q^3}\cdot\frac{1-q^{16}}{1-q^4}\cdots \\
&= 1 \big/ (1-q)(1-q^2)(1-q^3)(1-q^5)(1-q^6)(1-q^7)(1-q^9)(1-q^{10})\cdots \\
&= \frac{(1+q^2)(1+q^4)(1+q^6)(1+q^8)(1+q^{10})\cdots}{(1-q)(1-q^3)(1-q^5)(1-q^7)(1-q^9)\cdots}
\end{aligned}$$

where the first, third, and fourth expressions respectively generate numbers of partitions with no parts occurring more than thrice, of those with no part a multiple of 4, and of those with no even part repeated. For these last partitions there is a consequence of Theorem 2.1.

Theorem 3.1. *The numbers of odd and even partitions of n into parts with no even part repeated are equal, unless n is triangular, in which case they differ by one.*

Proof. We give a bijection, which preserves parity, between partitions in which no even part is repeated, and partitions in which no part occurs more than thrice. Given a partition with no even part repeated, we express the multiplicity of each odd part $2p+1$ as a sum of distinct powers of two, and for each such power 2^k with $k \geq 2$, we replace 2^k copies of $2p+1$ by 2 copies of $(2p+1)2^{k-1}$ to get a partition with no part occurring more than thrice. This mapping can clearly be inverted by changing any pair of repeated even parts $(2p+1)2^{k-1}$ to 2^k copies of $2p+1$. □

We have been unable to find a bijection for the partitions with no part a multiple of 4, and in any case the parities of these do not match those of the other partitions considered here.

Sequence (3.1) satisfies an analog of Euler's famous pentagonal number theorem. To present this it is convenient to recall Conway's generation of the pentagonal numbers from the triangular numbers [5, p. 96]. In Table 2,

triangular	0	1	3	6	10	15	21	28	36	45	55	66	78	$\cdots$
pentagonal	0		1	2		5	7		12	15		22	26	$\cdots$
sign	+		−	−		+	+		−	−		+	+	$\cdots$

Table 2.

the pentagonal numbers are one-third of the triangular ones (when that is an integer). They are generated in pairs $\frac{1}{2}k(3k \mp 1)$ of positive and negative rank. Signs + and − are allocated according as these pairs are of the same or opposite parity. Then Euler's theorem [1, p. 12, Corollary 1.8] can be stated in the easily remembered form

$$p(n-0) - p(n-1) - p(n-2) + p(n-5) + p(n-7) \\ - p(n-12) - p(n-15) + p(n-22) + - - + + \cdots = 0^n \tag{3.2}$$

where the right side is 1 if $n = 0$, and 0 otherwise. Our analog is as follows:

Theorem 3.2. *The numbers of partitions with parts occurring at most thrice satisfy equation (3.2) unless n is four times a pentagonal number. If $n = 2k(3k \mp 1)$, then the right side of (3.2) must be modified to $(-1)^k$.*

Proof. Note that $\prod_{i=1}^{\infty}(1-q^i)$ expands to

$$1 - q - q^2 + q^5 + q^7 - q^{12} - + + - - \cdots . \tag{3.3}$$

So our Euler-type sums are given by the coefficients of

$$\prod_{i=1}^{\infty}(1+q^i+q^{2i}+q^{3i})(1-q^i) = \prod_{i=1}^{\infty}(1-q^{4i})$$

which is (3.3) with q replaced by q^4. Thus the coefficient of q^n is zero unless n is 4 times a pentagonal number, when we get ± 1 with the same sign pattern as in Table 2. □

For example:

$$\begin{aligned} n &= 9 & 22 - 16 - 12 + 4 + 2 &= 0 \\ n &= 18 & 208 - 166 - 132 + 64 + 38 - 9 - 3 &= 0 \\ n &= 27 & 1280 - 1060 - 876 + 484 + 320 - 105 - 50 + 6 + 1 &= 0 \end{aligned}$$

etc., unless n is 4 times a pentagonal number, when we get a familiar pattern:

$$\begin{aligned} n &= 4 \times 1 & 4 - 3 - 2 &= -1 \\ n &= 4 \times 2 & 16 - 12 - 9 + 3 + 1 &= -1 \\ n &= 4 \times 5 & 320 - 258 - 208 + 105 + 64 - 16 - 6 &= +1 \\ n &= 4 \times 7 & 1539 - 1280 - 1060 + 592 + 395 - 132 - 64 + 9 + 2 &= +1 \end{aligned}$$

where the signs on the right are parallel with those in Euler's theorem.

4. A start(l)ing sequence

In Table 1, look at the top few values in each column, specifically the $t^{\pm}_{n,l}$ with $l = k$, that is, the entries of maximum weight. For $n = \frac{1}{2}k(k+1) + m$ with $0 \le m \le k$, the sequence

$$1/0, 1, 2, 4, 7, 12, 20, 32, 50, 77, 116, 172, 252, \ldots \tag{4.1}$$

emerges as k increases. We will denote this sequence by $u^{\pm}(m)$, where u^+ gives the values for partitions of the same parity as k, and u^- those for partitions of the opposite parity. These two values are equal unless $m = 0$, when $u^+(0) = 1$ and $u^-(0) = 0$.

Sequence (4.1) is A014968 in OEIS [17, on line, not in book]. In fact, we will prove the following two theorems in Section 12.

Theorem 4.1.

$$\begin{aligned}\sum_{m=0}^{\infty}[u^+(m) + u^-(m)]q^m &= \frac{1}{\vartheta_4(q)} \\ &= \frac{1}{(1-q)^2(1-q^2)(1-q^3)^2(1-q^4)(1-q^5)^2\cdots} \\ &= \prod_{n=1}^{\infty}\frac{1}{(1-q^n)(1-q^{2n-1})}\end{aligned}$$

where $\vartheta_4(q)$ is a Jacobi theta function [22, Chap. XXI].

Theorem 4.2. *$u^+(m)$ is odd just if m is a square.*

The sequence of coefficients in the expansion of $1/\vartheta_4$ is A015128 in OEIS, and is the Euler transform [17, p. 20] of the sequence $\{2, 1, 2, 1, \ldots\}$.

We next show, assuming Theorem 2.1 for now, that the first nonzero entries in each of the columns of Table 1 will indeed continue to yield the same sequence (4.1). For instance, the entries 12 12 for $n = 20$ with weight 3^5 occur again at $n = 26$ and weight 3^6, and then for $n = 33$, weight 3^7, and whenever n is later of shape $\frac{1}{2}(k^2 + k + 10)$, at weight 3^k.

Theorem 4.3. *If $n = \frac{1}{2}k(k+1) + m$ with $0 \le m \le k$, write $t^{\pm}_{n,k}$ as $v^{\pm}_{m,k}$. Then $v^+_{m,k} = v^+_{m,k+1}$ and $v^-_{m,k} = v^-_{m,k+1}$.*

Proof. We construct a specific bijection between maximum weight partitions of n and similar such partitions of $n + k + 1$ which have opposite parity. We will use two lemmas.

Lemma 4.4. *Maximum weight partitions contain no triple part.*

Proof. If a partition contains a triple part, we can increase its weight by adding one of the three equal parts to a largest part. □

Lemma 4.5. *If, in a maximum weight partition of n, all part sizes 1, 2, $\ldots$, $i-1$ occur, but i does not, then no part size $j > i$ can occur more than once.*

Proof. If a part size j occurs more than once, then we can get a partition of greater weight by replacing a part j by a part of size i, and adding $j - i$ to a largest part. □

To complete the proof of Theorem 4.3 we relate a partition of n of the type described in Lemma 4.5 to a partition of $n + k + 1$ by inserting a part i and increasing all larger parts by 1. Since each part size $1, 2, \dots, i - 1$ occurred at least once, i is 1 more than the number of part sizes less than i. That is, n increases by 1 more than the number of part sizes in the original partition. Because the original partition had maximum weight, its number of part sizes was k and the new partition is a partition of $n + 1 + k$ which has $k + 1$ part sizes and no triple parts and is thus of maximum weight.

To reverse the process, start with any partition of $n+k+1$ of (maximum) weight 3^{k+1} and suppose that all part sizes $1, 2, \dots, i$ occur, but that $i + 1$ does not. Since the weight is 3^{k+1}, i cannot be zero, else the sum is at least

$$(k+2) + \cdots + 3 + 2 = (k+1) + (k+1) + \cdots + 3 + 2 + 1 > n + k + 1.$$

As we have seen, no part $j > i + 1$ can occur twice. So we can omit part i and decrease any part of size $> i + 1$ by 1, and reduce the sum by the number $k + 1$ of part sizes, and obtain a partition of n of weight 3^k. □

The bijection constructed here will be generalized in Section 11. It is illustrated below by the twelve even partitions of 20 of weight 3^5 which are mapped onto the twelve odd partitions of $26 = 20 + 5 + 1$.

$$\begin{array}{lcll}
9\ 4\ 3\ 2\ 1^2 & \longrightarrow & 10\ 5\ 4\ 3\ 2\ 1^2 & (i=5)\\
8\ 5\ 3\ 2\ 1^2 & \longrightarrow & 9\ 6\ 4\ 3\ 2\ 1^2 & (i=4)\\
8\ 4\ 3\ 2^2\ 1 & \longrightarrow & 9\ 5\ 4\ 3\ 2^2\ 1 & (i=5)\\
7\ 6\ 3\ 2\ 1^2 & \longrightarrow & 8\ 7\ 4\ 3\ 2\ 1^2 & (i=4)\\
7\ 5\ 4\ 2\ 1^2 & \longrightarrow & 8\ 6\ 5\ 3\ 2\ 1^2 & (i=3)\\
7\ 5\ 3\ 2^2\ 1 & \longrightarrow & 8\ 6\ 4\ 3\ 2^2\ 1 & (i=4)\\
7\ 4\ 3^2\ 2\ 1 & \longrightarrow & 8\ 5\ 4\ 3^2\ 2\ 1 & (i=5)\\
6\ 5\ 4\ 3\ 1^2 & \longrightarrow & 7\ 6\ 5\ 4\ 2\ 1^2 & (i=2)\\
6\ 5\ 4\ 2^2\ 1 & \longrightarrow & 7\ 6\ 5\ 3\ 2^2\ 1 & (i=3)\\
6\ 5\ 3^2\ 2\ 1 & \longrightarrow & 7\ 6\ 4\ 3^2\ 2\ 1 & (i=4)\\
6\ 4^2\ 3\ 2\ 1 & \longrightarrow & 7\ 5\ 4^2\ 3\ 2\ 1 & (i=5)\\
5^2\ 4\ 3\ 2\ 1 & \longrightarrow & 6\ 5^2\ 4\ 3\ 2\ 1 & (i=6)
\end{array}$$

5. Partitions of triangular numbers

Table 3 comes from the bold-headed rows of Table 1. The entries $e_{k,l} = t^+_{n,l} - t^-_{n,l}$ are the differences of the numbers of even and odd partitions of $n = \frac{1}{2}k(k + 1)$ of weight 3^l.

Theorem 5.1. *The quantities $e_{k,l}$ satisfy the recurrence*

$$e_{k+1,l} = e_{k,l} - e_{k,l-1} - e_{k-1,l}. \tag{5.1}$$

		l																	
n	k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	1																	
1	1	0	–1																
3	2	–1	–1	1															
6	3	–1	1	2	–1														
10	4	0	3	0	–3	1													
15	5	1	2	–5	–2	4	–1												
21	6	1	–2	–7	6	5	–5	1											
28	7	0	–5	0	15	–5	–9	6	–1										
36	8	–1	–3	12	9	–25	1	14	–7	1									
45	9	–1	3	15	–18	–29	35	7	–20	8	–1								
55	10	0	7	0	–42	14	63	–42	–20	27	–9	1							
66	11	1	4	–22	–24	85	14	–112	42	39	–35	10	–1						
78	12	1	–4	–26	40	95	–134	–84	174	–30	–65	44	–11	1					
91	13	0	–9	0	90	–30	–243	162	216	–243	0	99	–54	12	–1				
105	14	–1	–5	35	50	–215	–79	489	–120	–429	308	55	–142	65	–13	1			
120	15	–1	5	40	–75	–235	379	406	–825	–66	737	–352	–143	195	–77	14	–1		
136	16	0	11	0	–165	55	693	–462	–1111	1188	495	–1144	351	273	–259	90	–15	1	
153	17	1	6	–51	–90	455	259	–1561	176	2365	–1430	–1287	1638	–273	–455	335	–104	16	–1

TABLE 3. Excess $e_{k,l}$ of the number of even partitions of weight 3^l over the number of such odd partitions, when n is the triangular number $\frac{1}{2}k(k+1)$.

We shall prove this in Section 10.

Note that with the conventions

$$\begin{cases} e_{0,-1} = 1,\ e_{k,l} = 0 & \text{for } k < l, \\ e_{k,-1} = 0 & \text{for } k > 0 \end{cases}$$

(compare Table 12 in Section 13) the recurrence (5.1) can be used to find all entries in Table 3 from the diagonal entries $e_{k,k} = (-1)^k$, which in turn correspond to the unique maximum weight partitions $1\,2\ldots\, k$ of the triangular numbers. Also, once (5.1) has been proved, we may multiply it by 3^l, sum over l, and then use induction on k to show that $\sum_{l\geq 0} 3^l e_{k,l} = (-1)^k(2k+1)$. That is,

$$\sum_{l\geq 0} 3^l(t^+_{n,l} - t^-_{n,l}) = (-1)^k(2k+1) \quad \text{for } n = \tfrac{1}{2}k(k+1)$$

as claimed in Theorem 2.1. For example, for the row in Table 3 with $n = 21$, $k = 6$ we see that

$$1 - 2\cdot 3 - 7\cdot 3^2 + 6\cdot 3^3 + 5\cdot 3^4 - 5\cdot 3^5 + 1\cdot 3^6 = 13 = (-1)^6(2\cdot 6 + 1).$$

6. Potholes, heights and phantoms

Our partitions, which have **p**arts **o**ccurring **th**rice **o**r **les**s, will take on several manifestations, each distinguished by an assignment of heights to the part sizes. We shall refer to these manifestations as **potholes**. We define the **total height** h of a partition to be the number of its multiplicities that are exactly 1 or 2. It is the base 3 logarithm of the weight.

Potholes will be written

$$\cdots\ 4^{m_4}_{h_4}\, 3^{m_3}_{h_3}\, 2^{m_2}_{h_2}\, 1^{m_1}_{h_1}$$

where m_i is the multiplicity of the part size i, with $0 \leq m_i \leq 3$, and h_i is the **height** of that part size. Parts with multiplicity 0 are usually, but not always, omitted.

Heights are assigned to the part sizes in every possible way subject to the conditions that $h_i = 0$ if m_i is 0 or 3, and $0 \leq h_i \leq 2$ if m_i is 1 or 2, and $\sum h_i \leq h$. The (**total**) **decrement** of a pothole is defined by $d = h - \sum h_i$. Decrements are also assigned individually to each multiplicity, $d_i = -h_i = 0$ if m_i is 0 or 3 and $d_i = 1 - h_i$ if m_i is 1 or 2, so that $d = \sum d_i$.

Each pothole with positive decrement, $d > 0$, is given an additional twin form by appending a **phantom zero part**, 0^{-1}_1, which, for compactness, we shall write as 0^-_1 or even as 0^-. The presence of a phantom always absorbs a unit of height, $h_0 = 1$, so that the decrement of any pothole which can accept a phantom must be greater than 0, whereas the decrement of any pothole which includes a phantom must be in the range $0 \leq d < h$. To keep $d = \sum d_i$, we assign a decrement $d_0 = -1$ to the phantom.

The parity and weight of a pothole are inherited from the partition from which it was derived. The parity is that of $\sum m_i$, the total number of parts. Note that, when a phantom is present, $m_0 = -1$ is to be included in the

decrement 0		$d = 1$		$d = 2$		weight
odd	even	odd	even	odd	even	
4_1^1	$4_0^1\,0_1^-$	4_0^1				3
$3_1^1\,1_0^1\,0_1^-$	$3_2^1\,1_0^1$	$3_0^1\,1_0^1\,0_1^-$	$3_1^1\,1_0^1$		$3_0^1\,1_0^1$	
$3_0^1\,1_1^1\,0_1^-$	$3_1^1\,1_1^1$		$3_0^1\,1_1^1$			3^2
	$3_0^1\,1_2^1$					
$2_0^2\,0_1^-$	2_1^2		2_0^2			3
$2_2^1\,1_0^2$	$2_1^1\,1_0^2\,0_1^-$	$2_1^1\,1_0^2$	$2_0^1\,1_0^2\,0_1^-$	$2_0^1\,1_0^2$		
$2_1^1\,1_1^2$	$2_0^1\,1_1^2\,0_1^-$	$2_0^1\,1_1^2$				3^2
$2_0^1\,1_2^2$						

TABLE 4.

sum, reversing the parity. The weight is 3^l, where l is the number of m_i that are 1 or 2; in particular, the phantom does not contribute to the weight.

Example. *The complete set of 24 potholes of 4 is listed in Table 4.*

Note that the total numbers of potholes of the partitions 4, 3 1, 2^2, $2\,1^2$ of 4 are respectively 3^1, 3^2, 3^1, 3^2. In general, we have

Theorem 6.1. *The number of potholes for a given partition of height h is 3^h.*

Proof. For any partition with total height h, the number of different potholes of decrement d which have no phantom part is the coefficient of x^{h-d} in $(1 + x + x^2)^h$ since each assignment of 0, 1, 2 to an h_i corresponds to a choice of term 1, x, x^2 from a factor $1 + x + x^2$. This equals the coefficient of x^{-d} in $(x^{-1} + 1 + x)^h$, or, by symmetry, the coefficient of x^d. We denote this **trinomial coefficient** of x^d by $\binom{h}{d}_3$.

On the other hand, the number of different potholes of decrement d which *do* have a phantom part is the coefficient of x^{d+1}, $\binom{h}{d+1}_3$, since the omission of the phantom part and its height 1 effectively increments the decrement. So the total number of potholes for a given partition is

$$\sum_{d=0}^{h} \binom{h}{d}_3 + \sum_{d=0}^{h-1} \binom{h}{d+1}_3 \quad \text{which equals} \quad \sum_{d=-h}^{h} \binom{h}{d}_3 \quad \text{by symmetry.}$$

Substituting $x = 1$ in $(x^{-1}+1+x)^h = \sum_{d=-h}^{h} \binom{h}{d}_3 x^d$ completes the proof. □

As a corollary, the total number of potholes for all partitions of n is the total of their unsigned weights. This is the coefficient of q^n in the expansion of $\prod_{i=1}^{\infty}(1 + q^i)^3$ (the left side of Jacobi's identity (1.1) with sign reversed); these coefficients form sequence A022568 in OEIS [17].

We shall define a combinatorial bijection between the even and the odd potholes of n which preserves the decrement, with a single exception when n is triangular.

7. Statures and shadows

We refer to the pair ${}^{m_i}_{h_i}$ as the **stature** of the part of size i. For positive part sizes, there are eight possible statures, which we classify as four corresponding pairs

$$\begin{array}{r|c|c|c|c|l} \textbf{small} & {}^{0}_{0} & {}^{1}_{0} & {}^{1}_{1} & {}^{2}_{0} & \quad m_i + h_i \le 2, \\ \textbf{great} & {}^{1}_{2} & {}^{2}_{1} & {}^{2}_{2} & {}^{3}_{0} & \quad m_i + h_i \ge 3. \end{array}$$

To bring a pothole into the form to which our bijection applies, we need to convert it to a **shadowed** form by replacing any part size i which has great stature by the one of corresponding small stature. This decreases the multiplicity by 1, and to maintain the sum of the parts, we introduce a **shadow**. This is a **negative** part $-i^{-1}$ which we often abbreviate to $(-i)^{-}$, taking the symbol $^{-}$ as a new stature for negative parts, unsubscripted because a shadow has no height. We extend this notation to the phantom part, shortening 0^{-}_{1} to 0^{-}. We also introduce the stature ^{0} with no subscript for absent nonpositive part sizes, so that positive and nonpositive parts have disjoint sets of statures (we say **positive** and **nonpositive** statures).

In writing potholes, we generally abbreviate $(-i)^{-}$ further to i^{-} in which the superscript serves the double purpose of negating the part size and of indicating the multiplicity -1, and we still tend to omit absent parts (of any sign).

For example, the partition $8^1\ 6^1\ 4^1\ 2^2\ 1^3$ of 25 has total height 4 (with part sizes 8, 6, 4, 2 contributing) so that one of its potholes is $8^1_0\ 6^1_0\ 4^1_2\ 2^2_2\ 1^3_0$. For this pothole the part sizes 4, 2 and 1 have great stature, so we rewrite it as

$$8^1_0\ 6^1_0\ 4^0_0\ 2^1_1\ 1^2_0\ 1^{-}\ 2^{-}\ 4^{-}$$

Here the part 4^0_0 is absent; only its shadow remains. Note that it is still true that $\sum i\, m_i = n$:

$$8(1) + 6(1) + 4(0) + 2(1) + 1(2) + (-1)(-1) + (-2)(-1) + (-4)(-1) = 25.$$

The total number of parts, counting the phantom part if it is present, of a shadowed pothole is $\sum |m_i|$. This number is not affected by shadowing.

We give each shadow i^{-} a decrement $d_{-i} = -1$, like the phantom. This accounts for the difference between the decrements of great positive statures and their small counterparts, so that $d = \sum d_i$ is preserved by shadowing.

In a shadowed pothole, the positive statures all have decrement 0 or 1, and the nonpositive statures have decrement -1 or 0. We separate these statures again into two classes, the statures ${}^{0}_{0}$, ${}^{1}_{1}$, $^{-}$ of **low decrement** and the statures ${}^{1}_{0}$, ${}^{2}_{0}$, ^{0} of **high decrement**, arranging each class in a column so that each horizontal pair has multiplicities differing by 1. We call the statures ${}^{1}_{1}$ and ${}^{2}_{0}$ **bold** and the statures ${}^{0}_{0}$ and ${}^{1}_{0}$ **normal**.

Table 5 summarizes the statures that occur in shadowed potholes.

decrement	$h_i = m_i$ low	$h_i < m_i$ high	
normal positive	$\substack{0\\0}$	$\substack{1\\0}$	$m_i + h_i < 2$
bold positive	$\substack{1\\1}$	$\substack{2\\0}$	$m_i + h_i = 2$
nonpositive	$-$	0	no height

TABLE 5. Categorizing shadowed statures.

Note that, since the part sizes occurring in any pothole are bounded, all statures of sufficiently large part size have low decrement and are normal, and all statures of sufficiently small part size have high decrement.

Also, the parity of a shadowed pothole can be found by adding (modulo 2) the number of bold statures of low decrement, the number of normal positive statures of high decrement, and the number of nonpositive statures of low decrement. Equivalently, we can add the number of all bold statures, the number of all positive statures of high decrement, and the number of nonpositive statures of low decrement.

8. THE BIJECTION

Our bijection operates on shadowed potholes by manipulating the statures of various part sizes. To this end we define two injective partial functions f_a and f_b on statures, which act as follows.

In the terminology of the last section, f_a replaces a stature of low decrement with its counterpart of high decrement, and f_b changes normal statures to bold ones.

f_a **a**dds to the decrement.
f_b makes the stature **b**old.

$$f_a: \ \substack{0\\0} \mapsto \substack{1\\0} \qquad \substack{1\\1} \mapsto \substack{2\\0} \qquad - \mapsto 0$$

$$f_b: \ \substack{0\\0} \mapsto \substack{1\\1} \qquad \substack{1\\0} \mapsto \substack{2\\0}$$

$$\begin{array}{ccc} & \xrightarrow{f_a} & \\ \substack{0\\0} & & \substack{1\\0} \\ f_b \downarrow & & \uparrow f_b^{-1} \\ \substack{1\\1} & & \substack{2\\0} \\ & & \\ - & & 0 \\ & \xleftarrow[f_a^{-1}]{} & \end{array}$$

FIGURE 1. The action of f_a and f_b: left, as lists of pairs; right, in the arrangement of Table 5.

Because f_a and f_b commute as partial functions, we may apply both of them in indiscriminate order to any stature on which they're both defined; this remains true if either or both are inverted.

Suppose that the least (rightmost) part size whose stature has low decrement is x and the least part size greater than x whose stature has high decrement is y, so that the rightmost string of low-decrement statures has part sizes i in the interval $y > i \geq x$. Suppose further that the least (necessarily positive) part size with bold stature is z, so that the rightmost string of positive part sizes i with normal stature is $z > i > 0$. We will be comparing the lengths $y - x$ and $z - 1$ of these two strings. It may happen that there are no statures of high decrement greater than x, or no positive parts with bold stature, yielding infinite strings, in which cases we let y and/or z be infinite.

If they are both infinite then there can be no shadow or phantom, otherwise the total decrement would be negative. In this case all positive part sizes have normal stature, and statures of low decrement are exhibited by part sizes running from ∞ down to $x = k + 1$ for some nonnegative k. That is, the only parts present are $k, k-1, \ldots, 1$; each has stature ${}^1_0$ and we have the pothole $k^1_0 \; \cdots \; 3^1_0\, 2^1_0\, 1^1_0$ of the triangular number $\frac{1}{2}k(k+1)$. It has height k, decrement k and the same parity as k. This is the only case in which we cannot compare the lengths of the strings, and the bijection fails.

For the rest of this section, we'll disregard this exception. Then y and z are not both infinite, and we have either

$$\textbf{Case 1: } y - x \leq z - 1 \qquad \text{or} \qquad \textbf{Case 2: } y - x > z - 1.$$

In Case 1 we carry out the bijection by applying f_a to part size x, f_a^{-1} to part size y, and f_b to part size $y - x$. In Case 2 we apply f_a^{-1} to part size $x - 1$, f_a to part size $z + x - 1$, and f_b^{-1} to part size z.

Here are a few examples. For the pothole 2^3_0 of 6, the shadowed form is $2^2_0\, 2^-$, so we have $x = -2$, $y = -1$, and $z = 2$. We are therefore in Case 1, and applying f_a to -2, f_a^{-1} to -1, and f_b to 1 we get $2^2_0\, 1^1_1\, 1^-$, which is then rewritten as $2^2_0\, 1^2_2$. Conversely, if we started with $2^2_0\, 1^2_2$, we would first rewrite it as $2^2_0\, 1^1_1\, 1^-$ and then have $x = -1$, $y = 0$, $z = 1$. We would then be in Case 2, and we would apply f_a^{-1} to -2, f_a to -1, and f_b^{-1} to 1 to get $2^2_0\, 1^0_0\, 2^-$ or 2^3_0.

For the pothole $6^1_0\, 2^1_0\, 1^1_0$ of 9, we have $x = 3$, $y = 6$, and (because there are no bold statures) $z = \infty$. Thus we are in Case 1; we must apply both f_a and f_b to $x = 3$ and $y - x = 3$ and f_a^{-1} to $y = 6$. This yields $6^0_0\, 3^2_0\, 2^1_0\, 1^1_0$ or $3^2_0\, 2^1_0\, 1^1_0$. On the other hand, if we had started with this last pothole, we would have $x = 4$, $y = \infty$, $z = 3$, so we would be in Case 2 and apply both f_a^{-1} and f_b^{-1} to $x - 1 = 3$ and $z = 3$ and f_a to $z + x - 1 = 6$ to get back to $6^1_0\, 2^1_0\, 1^1_0$.

For an example involving the phantom, consider the pothole $3^2_0\; 2^1_0\; 0^-$ of 8. This has $x = 0$, $y = 2$, $z = 3$, so we are in Case 1; the bijection applies f_a to 0 and both f_a^{-1} and f_b to 2, resulting in $3^2_0\; 2^1_1$. Going the other way, for the pothole $3_0^2\ 2_1^1$ we have $x = 1$, $y = 3$, $z = 2$, so we are in Case 2; to carry out the bijection we apply f_a^{-1} to 0 (which makes the phantom appear) and both f_a and f_b^{-1} to 2, resulting in $3_0^2\ 2_0^1\ 0^-$.

As a final example, which you can follow along in Table 6, consider the pothole $8_0^1\ 6_0^1\ 4_2^1\ 2_2^2\ 1_0^3$ of 25. In the last section, we found the shadowed form $8_0^1\ 6_0^1\ 2_1^1\ 1_0^2\ 1^-\ 2^-\ 4^-$. We see that $x = -4$, $y = -3$, and $z = 1$, so we are in Case 2 and we should apply f_a^{-1} to -5, f_a to -4, and f_b^{-1} to 1. This yields $8_0^1\ 6_0^1\ 2_1^1\ 1_0^1\ 1^-\ 2^-\ 5^-$, which is then rewritten as $8_0^1\ 6_0^1\ 5_2^1\ 2_2^2\ 1_1^2$. Again, if we had started with the latter pothole, we would have found $x = -5$, $y = -4$, $z = 2$ from the shadowed form, which would have put us in Case 1. Applying f_a to -5, f_a^{-1} to -4, and f_b to $y - x = 1$ would have changed $8_0^1\ 6_0^1\ 2_1^1\ 1_0^1\ 1^-\ 2^-\ 5^-$ to $8_0^1\ 6_0^1\ 2_1^1\ 1_0^2\ 1^-\ 2^-\ 4^-$, the shadowed form of our original pothole.

As in these examples, it is straightforward to check in general that the bijection takes Case 1 potholes to Case 2 potholes and vice versa, and that it is its own inverse. For instance, if we're in Case 1, then carrying out the bijection will make $y - x$ bold, so if we denote the values of x, y, z for the new pothole by x', y', z' we will have $z' \le y - x$. We also have $x' \ge x + 1$ since x now has high decrement, $y' \ge y + 1$ since y now has low decrement, and $y' - x' \ge y - x$ since any parts between x and y have unchanged (low) decrement. So $y' - x' \ge z'$, that is, $y' - x' > z' - 1$, and the new pothole is in Case 2.

In Table 7, we exhibit the entire bijection for $n = 6$.

Table 8 summarizes Table 7. Its nine rows are the partitions of 6 which satisfy our conditions, and its columns are the parities, the partitions, the numbers of odd and even potholes of successively decremented height, and the signed weight of the partition.

The total numbers of odd and even potholes are the same, except for the unique pothole of maximum weight. The total signed weight of all partitions is $(-1)^k(2k + 1)$ for $k = 3$, as desired.

In general, to show that our bijection does what it ought and carries potholes of n to potholes of n of opposite parity, we must check that it is well behaved with respect not only to parity and the total of the parts (i.e. n), but also to the decrement, in the sense that the condition $\sum h_i \le h$ on potholes remains satisfied. In fact, the bijection will leave the decrement $d = h - \sum h_i$ unchanged. It's enough to show all of this for only one direction of the bijection; we pick Case 1. Taking the three properties in turn:

$n = \sum i m_i$ *is preserved.* Application of either f_a or f_b to part size k increments the multiplicity of parts of size k if $k > 0$, but decrements the multiplicity of parts of size $-k$ if $k \le 0$. In either case $\sum i m_i$ is increased by k. So when we apply f_a to part size x, f_a^{-1} to part size y, and f_b to part size $y - x$, the sum $n = \sum i m_i$ is increased by $x - y + (y - x) = 0$, so it is unchanged.

⋯	9	8	7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	⋯	
⋯	$^0_0$	$^1_0$	$^0_0$	$^1_0$	$^0_0$	$^1_2$	$^0_0$	$^2_2$	$^3_0$									unshadowed form
						g		g	g									great statures
⋯	$^0_0$	$^1_0$	$^0_0$	$^1_0$	$^0_0$	$^0_0$	$^0_0$	$^1_1$	$^2_0$	0	–	–	0	–	0	0	⋯	shadowed form
⋯	l	h	l	h	l	l	l	l	h	h	l	l	h	l	h	h	⋯	decrements
													y	x				
								b	b									bold statures
									z				y	x				$y-x=1\geq z$
									z					$(x-1+z)$	$(x-1)$			
									$\downarrow f_b^{-1}$					$\downarrow f_a$	$\downarrow f_a^{-1}$			
⋯	$^0_0$	$^1_0$	$^0_0$	$^1_0$	$^0_0$	$^0_0$	$^0_0$	$^1_1$	$^1_0$	0	–	–	0	0	–	0	⋯	shadowed form
⋯	$^0_0$	$^1_0$	$^0_0$	$^1_0$	$^1_2$	$^0_0$	$^0_0$	$^2_2$	$^2_1$									unshadowed form

TABLE 6. The bijection sends the pothole $8^1_0\ 6^1_0\ 4^1_2\ 2^2_2\ 1^3_0$ of 25 to $8^1_0\ 6^1_0\ 5^1_2\ 2^2_2\ 1^2_1$ and vice versa.

$d=0$			$d=1$			$d=2$			$d=3$
6^1_1	$\longleftrightarrow$	$6^1_0 0^-_1$	6^1_0	$\longleftrightarrow$	$5^1_1 1^1_0$				
$5^1_1 1^1_0 0^-_1$	$\longleftrightarrow$	$5^1_1 1^1_1$							
$5^1_0 1^1_1 0^-_1$	$\longleftrightarrow$	$5^1_0 1^1_2$	$5^1_0 1^1_0 0^-_1$	$\longleftrightarrow$	$5^1_0 1^1_1$				
$4^1_1 2^1_0 0^-_1$	$\longleftrightarrow$	$4^1_1 2^1_1$							
$4^1_0 2^1_1 0^-_1$	$\longleftrightarrow$	$4^1_0 1^2_1 0^-_1$	$4^1_0 2^1_0 0^-_1$	$\longleftrightarrow$	$4^1_0 2^1_1$				
$4^1_2 1^2_0$	$\longleftrightarrow$	$5^1_2 1^1_0$							
$4^1_1 1^2_1$	$\longleftrightarrow$	$4^1_1 1^2_0 0^-_1$	$4^1_0 1^2_1$	$\longleftrightarrow$	$4^1_0 1^2_0 0^-_1$	$4^1_0 1^2_0$	$\longleftrightarrow$	$4^1_0 2^1_0$	
$4^1_0 1^2_2$	$\longleftrightarrow$	$4^1_0 2^1_2$	$4^1_1 1^2_0$	$\longleftrightarrow$	$4^1_1 2^1_0$				
$3^2_0 0^-_1$	$\longleftrightarrow$	$3^1_0 2^1_0 1^1_2 0^-_1$							
$3^1_2 2^1_1 1^1_0$	$\longleftrightarrow$	$2^2_2 1^2_0$	$3^1_2 2^1_0 1^1_0$	$\longleftrightarrow$	$2^2_1 1^2_0$	$3^1_1 2^1_0 1^1_0$	$\longleftrightarrow$	$5^1_0 1^1_0$	$\boxed{3^1_0 2^1_0 1^1_0}$
$3^1_1 2^1_2 1^1_0$	$\longleftrightarrow$	$3^1_1 1^3_0$	$3^1_1 2^1_1 1^1_0$	$\longleftrightarrow$	3^2_0	$3^1_0 2^1_1 1^1_0$	$\longleftrightarrow$	$2^2_0 1^2_0$	
$3^1_2 2^1_0 1^1_1$	$\longleftrightarrow$	$4^1_2 2^1_0$	$3^1_0 2^1_2 1^1_0$	$\longleftrightarrow$	$3^1_0 1^3_0$	$3^1_0 2^1_0 1^1_1$	$\longleftrightarrow$	$3^1_0 2^1_0 1^1_0 0^-_1$	
$3^1_1 2^1_1 1^1_1$	$\longleftrightarrow$	$3^1_1 2^1_1 1^1_0 0^-_1$	$3^1_1 2^1_0 1^1_1$	$\longleftrightarrow$	$3^1_1 2^1_0 1^1_0 0^-_1$				
$3^1_0 2^1_2 1^1_1$	$\longleftrightarrow$	3^2_1	$3^1_0 2^1_1 1^1_1$	$\longleftrightarrow$	$3^1_0 2^1_1 1^1_0 0^-_1$				
$3^1_1 2^1_0 1^1_2$	$\longleftrightarrow$	$3^1_1 2^1_0 1^1_1 0^-_1$	$3^1_0 2^1_0 1^1_2$	$\longleftrightarrow$	$3^1_0 2^1_0 1^1_1 0^-_1$				
$3^1_0 2^1_1 1^1_2$	$\longleftrightarrow$	$3^1_0 2^1_1 1^1_1 0^-_1$							
$3^1_0 1^3_0 0^-_1$	$\longleftrightarrow$	$3^1_0 2^1_2 1^1_0 0^-_1$							
2^3_0	$\longleftrightarrow$	$2^2_0 1^2_2$							
$2^2_1 1^2_0 0^-_1$	$\longleftrightarrow$	$3^1_2 2^1_0 1^1_0 0^-_1$	$2^2_0 1^2_0 0^-_1$	$\longleftrightarrow$	$2^2_0 1^2_1$				
$2^2_0 1^2_1 0^-_1$	$\longleftrightarrow$	$2^2_1 1^2_1$							

TABLE 7. The bijection for potholes of 6, which fails for $3^1_0 2^1_0 1^1_0$.

		$d=0$		$d=1$		$d=2$		$d=3$	signed
		$-$	$+$	$-$	$+$	$-$	$+$	$-$	weight
$-$	6	1	1	1					-3
$+$	$5\,1$	2	3	1	2		1		9
$+$	$4\,2$	2	3	1	2		1		9
$-$	$4\,1^2$	3	2	2	1	1			-9
$+$	3^2	1	1		1				3
$-$	$3\,2\,1$	7	6	6	3	3	1	1	-27
$+$	$3\,1^3$	1	1		1				3
$-$	2^3	1							-1
$+$	$2^2 1^2$	2	3	1	2		1		9
Totals		-20	20	-12	12	-4	4	-1	-7

TABLE 8. Numbers of potholes of 6 of decreasing height, with total weight.

Decrement is preserved. Applying f_a adds 1 to the decrement of the part size to which it applies and therefore to the total decrement; this is undone by the application of f_a^{-1}. Applying f_b leaves the decrement alone, so the total change in decrement is $1 - 1 + 0 = 0$.

Parity is reversed. Both f_a and f_b reverse parity, since they either increment or decrement multiplicity. So the overall effect of f_a and f_a^{-1} and f_b is to reverse parity.

We remark that the bijection we have just presented is actually one of a pair of complementary bijections. To present the other bijection we introduce an involution *sp* on partitions with heights assigned and a possible phantom, that is defined on (unshadowed) partitions by negating the decrement of every part, and by introducing an absent phantom or deleting a present one. That is, among positive statures *sp* exchanges $^1_0$ and $^1_2$ and exchanges $^2_0$ and $^2_2$, leaving the other four statures unchanged; among negative statures (which at this stage only occur on 0) it exchanges ^{0} and $^-$. Since *sp* negates the decrement of each nonzero part size and swaps decrements -1 and 0 associated to the presence or absence of the phantom, its action on the overall decrement is to replace d by $-1-d$. It therefore sends no potholes to potholes; in fact, among all partitions with assigned heights and possible phantoms, it exchanges those which are potholes with those which aren't. However, *sp* preserves parity and the quantity $n = \sum im_i$ on all partitions with heights.

For partitions, with heights assigned, that are in shadowed form, the effect of *sp* on statures can be described easily. In fact, if P is such a partition, the sets of positive part sizes of P and of $sp(P)$ that have bold stature are identical, and any part size k has low decrement in P if and only if $-k$ has high decrement in $sp(P)$. For example, for $P = 6^1_0\ 2^1_1\ 1^1_0\ 1^-\ 2^-\ 5^-$, only 2 has bold stature, the only positive parts with high decrement are 6 and 1, and the only negative parts with low decrement are -1, -2, and -5. To find $sp(P)$, we first pass to the "unshadowed" form of P, which is $6^1_0\ 5^1_2\ 2^2_2\ 1^2_1$, and then change statures and add a phantom to get $6^1_2\ 5^1_0\ 2^2_0\ 1^2_1\ 0^-$. This yields the shadowed form $sp(P) = 6^0_0\ 5^1_0\ 2^2_0\ 1^1_0\ 0^-\ 1^-\ 6^-$, in which the positive parts with high decrement are 1, 2, and 5 while the negative parts with low decrement are -1 and -6. In both P and $sp(P)$, only 2 has bold stature. In this example, $sp(P)$ is a pothole, while P is not.

The complementary bijection is simply the conjugate of our main bijection by *sp*: it acts by applying *sp*, then our main bijection, then *sp* again. We will invoke this complementary bijection in Section 9.

9. The maximum weight case

In this section we examine the bijection in more detail when h is as large as possible and $d = h$. There are then no phantoms and, by Lemma 4.4, the only possible statures are $^0_0$ (for absent parts), $^1_0$ and $^2_0$. In particular, each maximum weight partition has a unique pothole of this type. If n is triangular, there is just one maximum weight partition, and it produces the exceptional pothole.

For non-triangular numbers $n = 2, 4, 5, 7, \ldots$ we get a bijection, which is shown in Table 9, on partitions of maximum weight.

$n=2$
$2 \longleftrightarrow 1^2$

$n=4$
$3\,1 \longleftrightarrow 2\,1^2$

$n=5$
$4\,1 \longleftrightarrow 2^2\,1$
$3\,2 \longleftrightarrow 3\,1^2$

$n=7$
$4\,2\,1 \longleftrightarrow 3\,2\,1^2$

$n=8$
$5\,2\,1 \longleftrightarrow 3\,2^2\,1$
$4\,3\,1 \longleftrightarrow 4\,2\,1^2$

$n=9$
$6\,2\,1 \longleftrightarrow 3^2\,2\,1$
$5\,3\,1 \longleftrightarrow 5\,2\,1^2$
$4\,3\,2 \longleftrightarrow 4\,3\,1^2$
$4\,2^2\,1 \longleftrightarrow 3\,2^2\,1^2$

$n=11$
$5\,3\,2\,1 \longleftrightarrow 4\,3\,2\,1^2$

$n=12$
$6\,3\,2\,1 \longleftrightarrow 4\,3\,2^2\,1$
$5\,4\,2\,1 \longleftrightarrow 5\,3\,2\,1^2$

$n=13$
$7\,3\,2\,1 \longleftrightarrow 4\,3^2\,2\,1$
$6\,4\,2\,1 \longleftrightarrow 6\,3\,2\,1^2$
$5\,4\,3\,1 \longleftrightarrow 5\,4\,2\,1^2$
$5\,3\,2^2\,1 \longleftrightarrow 4\,3\,2^2\,1^2$

$n=14$
$8\,3\,2\,1 \longleftrightarrow 4^2\,3\,2\,1$
$7\,4\,2\,1 \longleftrightarrow 7\,3\,2\,1^2$
$6\,5\,2\,1 \longleftrightarrow 6\,3\,2^2\,1$
$6\,4\,3\,1 \longleftrightarrow 6\,4\,2\,1^2$
$5\,4\,3\,2 \longleftrightarrow 5\,4\,3\,1^2$
$5\,4\,2^2\,1 \longleftrightarrow 5\,3\,2^2\,1^2$
$5\,3^2\,2\,1 \longleftrightarrow 4\,3^2\,2\,1^2$

$n=16$
$6\,4\,3\,2\,1 \longleftrightarrow 5\,4\,3\,2\,1^2$

$n=17$
$7\,4\,3\,2\,1 \longleftrightarrow 5\,4\,3\,2^2\,1$
$6\,5\,3\,2\,1 \longleftrightarrow 6\,4\,3\,2\,1^2$

$n=18$
$8\,4\,3\,2\,1 \longleftrightarrow 5\,4\,3^2\,2\,1$
$7\,5\,3\,2\,1 \longleftrightarrow 7\,4\,3\,2\,1^2$
$6\,5\,4\,2\,1 \longleftrightarrow 6\,5\,3\,2\,1^2$
$6\,4\,3\,2^2\,1 \longleftrightarrow 5\,4\,3\,2^2\,1^2$

TABLE 9. Bijections on partitions of maximum weight for non-triangular n.

We will now develop a description of this bijection without explicit use of potholes. Recall from Table 5 that, of the three occurring statures, only $\genfrac{}{}{0pt}{1}{0}{0}$ has low decrement and only $\genfrac{}{}{0pt}{1}{2}{0}$ is bold. Thus in the notation of Section 8, x and y are, respectively, the least part of the least string of missing parts and the least part greater than this string, which is of length $y-x$. Also, z is the least repeated part. From Lemma 4.5, no repeated part can be greater than x, so either $z < x$ or $z = \infty$.

Now if $y - x \le z - 1$, then y must be finite and the bijection will change the statures of part sizes x, y and $y-x$. The stature of part x changes from $\genfrac{}{}{0pt}{1}{0}{0}$ (via f_a) to $\genfrac{}{}{0pt}{1}{1}{0}$. Part size y must have stature $\genfrac{}{}{0pt}{1}{1}{0}$ since it must occur and cannot repeat, so the bijection changes the stature of this part (via f_a^{-1}) to $\genfrac{}{}{0pt}{1}{0}{0}$. If z is finite, then $y - x \le z - 1 < x$, so the part $y-x$ cannot be missing or repeated and column $y-x$ must have stature $\genfrac{}{}{0pt}{1}{1}{0}$. Its stature then changes (via f_b) to $\genfrac{}{}{0pt}{1}{2}{0}$. On the other hand, if $z = \infty$ there are two cases. If

$y-x<x$, the bijection acts as before, while if $y-x=x$, the stature of the part x changes (via f_a and f_b) all the way from $\begin{smallmatrix}0\\0\end{smallmatrix}$ to $\begin{smallmatrix}2\\0\end{smallmatrix}$. The case $y-x>x$ is impossible because $y-x \le y-1$ would be a missing part and we could get a partition of greater weight by replacing the part y by separate parts $y-x$ and x.

So far we have seen that whenever $y-x<z$ the bijection acts on the partition by splitting up the part y (the first part *greater* than the least string of missing parts) into parts x and $y-x$. We now consider the case $y-x \ge z$, in which the bijection affects parts $x-1$, z and $x-1+z$. As z is finite, z is a repeated part and the bijection changes its stature $\begin{smallmatrix}2\\0\end{smallmatrix}$ (via f_b^{-1}) to $\begin{smallmatrix}1\\0\end{smallmatrix}$. Because $x \le x-1+z \le y-1$, $x-1+z$ is a missing part and its stature changes from $\begin{smallmatrix}0\\0\end{smallmatrix}$ (via f_a) to $\begin{smallmatrix}1\\0\end{smallmatrix}$. The part $x-1$ cannot be missing, but there are three possibilities. If $x-1<z$, then $x-1$ occurs just once and so the bijection changes its stature from $\begin{smallmatrix}1\\0\end{smallmatrix}$ (via f_a^{-1}) to $\begin{smallmatrix}0\\0\end{smallmatrix}$. If $x-1=z$, its stature is $\begin{smallmatrix}2\\0\end{smallmatrix}$ and changes to $\begin{smallmatrix}0\\0\end{smallmatrix}$ by $f_a^{-1} \circ f_b^{-1}$. If $x-1>z$, the part $x-1$ cannot be repeated because otherwise we could get a partition of greater weight by replacing one part $x-1$ and one part z with a new part $x-1+z$. So just as when $x-1<z$, the stature of $x-1$ changes from $\begin{smallmatrix}1\\0\end{smallmatrix}$ to $\begin{smallmatrix}0\\0\end{smallmatrix}$. In all three variations the bijection combines the first repeated part z with the part $x-1$ to form a new part $x-1+z$.

In summary, we have the following description of the bijection on partitions of maximum weight for nontriangular numbers n: find the least string of consecutive missing parts; say it has length s. Also find the smallest repeated part r. If $r \le s$ (in particular, if the only parts occurring are consecutive, 1,2,…,k, so that $s=\infty$), add r to the part just below the string of missing parts. If $r>s$ (in particular if there is no repeated part, so that $r=\infty$) take the first part y above the string of missing parts and split it into parts $(y-s)$ and s.

As mentioned at the end of Section 8, there is a second bijection, related to the first by conjugation by the involution *sp*. For the case when $d=h$ and h is as large as possible, this second bijection also induces a bijection on partitions of maximum weight, provided n is not triangular. We give an example to illustrate this; a general description will follow.

Take the partition $6\,5\,3^2\,2\,1$ of 20, of maximum weight 3^5. The associated pothole with $d=h=5$ is $6_0^1\,5_0^1\,3_0^2\,2_0^1\,1_0^1$.

We apply *sp* to get $6_2^1\,5_2^1\,3_2^2\,2_2^1\,1_2^1\,0^-$ with $d=-5$, which, when rewritten in shadowed form, is $3_1^1\,0^-\,1^-\,2^-\,3^-\,5^-\,6^-$.

The least string of low decrements starts at -6. We have $x=-6$, $y=-4$, $z=3$. Since $y-x \le z-1$ we apply f_a to -6, f_a^{-1} to -4, and f_b to 2, transforming the partition to $3_1^1\,2_1^1\,0^-\,1^-\,2^-\,3^-\,4^-\,5^-$. Recombining shadows with positive parts we get $5_2^1\,4_2^1\,3_2^2\,2_2^2\,1_2^1\,0^-$. Finally we apply *sp* again and discard heights to arrive at $5\,4\,3^2\,2^2\,1$.

For any partition of maximum weight, after the first application of *sp*, x will be the negative of the largest part of the partition. The least string of consecutive low decrement parts corresponds to the greatest string of high decrement parts before the spin, and therefore to the string of largest consecutive part sizes (*italic* in the examples below). Call the length of that string the **slope**, s, of the partition. This is consistent with the terminology used in describing Franklin's famous bijection [20, p. 158, Theorem 15.5] for partitions with distinct parts. For example, for the partition $\mathit{6}\,\mathit{5}\,3^2\,2\,1$, $s=2$, while for the partition $\mathit{5}\,\mathit{4}\,\mathit{3}^2\,\mathit{2}^2\,\mathit{1}$, $s=5$.

To reiterate, after the first application of sp we have $y-x=s$.

Since which part sizes have normal stature is unaffected by sp, z will still be the least repeated part. If $s=y-x<z$, then, as in the example above, the effect of the bijection will be to split the largest part of the partition into two parts, one of which is the largest missing part size. In our example $6\,5\,3^2\,2\,1$, the largest missing size is 4, and 6 is replaced by $4+2$. Conversely, if $s\ge z$, then a new part, larger by 1 than the original largest part, is formed by combining the first repeated part z with another part of appropriate size. For example, for the partition $6\,5\,3\,2\,1^2$ with $z=1$, $s=2$, we form a new part 7 by combining 1 and 6, so the image of this partition under the second bijection is $7\,5\,3\,2\,1$.

For weights less than the maximum, neither of our two bijections seems to have a straightforward generalization to a direct bijection between partitions rather than potholes. Such a direct bijection does exist for minimum weight $3^0=1$. Partitions of weight 1 only occur for $n=3m$, since every part must occur thrice. The bijection is then obtained from Franklin's bijection by repeating each part thrice. This breaks down if and only if m is a pentagonal number, which, as we saw in Section 3, is exactly if n is a triangular number divisible by 3.

So we have only shown that $t^-_{n,l}=t^+_{n,l}$ for nontriangular n and maximal l (or $l=0$). In the next section we use a count of potholes to extend this result to arbitrary l.

10. Proofs of Theorems 2.1 and 5.1

Let $n=\frac{1}{2}k(k+1)+m$ with $0<m\le k$, so that n is not a triangular number. Now that we have a bijection which preserves decrements and reverses parity on the set of all potholes of n, we are in a position to prove the first part of Theorem 2.1, that $t^+_{n,l}=t^-_{n,l}$ for every weight 3^l. To do so we consider the numbers $p^{\pm}_{n,d}$ of even and odd potholes of decrement d which correspond to partitions of n.

We noted in the proof of Theorem 6.1 that potholes can be counted by trinomial coefficients. Specifically, for a given partition with l part sizes that occur just once or twice, there are $\binom{l}{d}_3$ potholes of decrement d without a phantom part and $\binom{l}{d+1}_3$ potholes of decrement d that do include a phantom.

Since the inclusion of a phantom part changes the parity of the pothole, we have

$$\begin{aligned} p^+_{n,d} &= t^+_{n,d} + \binom{d+1}{d}_3 t^+_{n,d+1} + \binom{d+2}{d}_3 t^+_{n,d+2} + \cdots \\ &\qquad\qquad + t^-_{n,d+1} + \binom{d+2}{d+1}_3 t^-_{n,d+2} + \cdots \\ &= \sum_{l=d}^{k} \binom{l}{d}_3 t^+_{n,l} + \sum_{l=d+1}^{k} \binom{l}{d+1}_3 t^-_{n,l} \end{aligned} \tag{10.1}$$

Similarly,

$$p^-_{n,d} = \sum_{l=d}^{k} \binom{l}{d}_3 t^-_{n,l} + \sum_{l=d+1}^{k} \binom{l}{d+1}_3 t^+_{n,l} \tag{10.2}$$

Our bijection shows that $p^+_{n,d} = p^-_{n,d}$ for all d. To show that $t^+_{n,l} = t^-_{n,l}$ for all l, we start with $l = k$ and work our way down using (10.1) and (10.2). For the maximal case $l = k$, we know from Section 9 that $t^+_{n,l} = t^-_{n,l}$. Now suppose that $t^+_{n,l} = t^-_{n,l}$ for all $l > d$. Then for decrement d we have

$$\begin{aligned} t^+_{n,d} &= p^+_{n,d} - \sum_{l=d+1}^{k} \binom{l}{d}_3 t^+_{n,l} - \sum_{l=d+1}^{k} \binom{l}{d+1}_3 t^-_{n,l} \\ &= p^-_{n,d} - \sum_{l=d+1}^{k} \binom{l}{d}_3 t^-_{n,l} - \sum_{l=d+1}^{k} \binom{l}{d+1}_3 t^+_{n,l} \\ &= t^-_{n,d} \end{aligned}$$

as claimed.

Now suppose that $n = \frac{1}{2}k(k+1)$ is a triangular number. Much of the previous argument will still apply, but because of the exceptional pothole $k^1_0 \cdots 2^1_0 1^1_0$ we now have

$$e_{k,k} = t^+_{n,k} - t^-_{n,k} = p^+_{n,k} - p^-_{n,k} = (-1)^k/0!$$

$$\begin{aligned} e_{k,k-1} = t^+_{n,k-1} - t^-_{n,k-1} &= p^+_{n,k-1} - \binom{k}{k-1}_3 t^+_{n,k} - t^-_{n,k} \\ &\qquad - p^-_{n,k-1} + \binom{k}{k-1}_3 t^-_{n,k} + t^+_{n,k} \\ &= \left[-\binom{k}{k-1}_3 + 1\right](t^+_{n,k} - t^-_{n,k}) \\ &= (-1)^{k-1}(k-1)/1! \end{aligned}$$

$$\begin{aligned}
e_{k,k-2} = t^+_{n,k-2} - t^-_{n,k-2} \quad = \quad & p^+_{n,k-2} - \binom{k-1}{k-2}_3 t^+_{n,k-1} - \binom{k}{k-2}_3 t^+_{n,k} \\
& - \binom{k-1}{k-1}_3 t^-_{n,k-1} - \binom{k}{k-1}_3 t^-_{n,k} \\
& - p^-_{n,k-2} + \binom{k-1}{k-2}_3 t^-_{n,k-1} + \binom{k}{k-2}_3 t^-_{n,k} \\
& + \binom{k-1}{k-1}_3 t^+_{n,k-1} + \binom{k}{k-1}_3 t^+_{n,k}
\end{aligned}$$

$$\begin{aligned}
= \quad & \left[-\binom{k-1}{k-2}_3 + \binom{k-1}{k-1}_3\right](t^+_{n,k-1} - t^-_{n,k-1}) \\
& + \left[-\binom{k}{k-2}_3 + \binom{k}{k-1}_3\right](t^+_{n,k} - t^-_{n,k}) \\
= \quad & (-1)^{k-2}(k-1)(k-4)/2!
\end{aligned}$$

and more generally, for $0 \le l < k$,

$$e_{k,l} = \sum_{j=l+1}^{k} \left[-\binom{j}{l}_3 + \binom{j}{l+1}_3\right] e_{k,j} \tag{10.3}$$

Note that (10.3) expresses each entry in Table 3 in terms of entries that are further right in the same row.

We now prove Theorem 5.1, that

$$e_{k+1,l} = e_{k,l} - e_{k,l-1} - e_{k-1,l}, \tag{5.1}$$

by induction on $k-l$. For $k-l = -1$, (5.1) reduces to $e_{k+1,k+1} = -e_{k,k}$ by our convention that $e_{k,l} = 0$ for $k < l$. This is already clear from $e_{k,k} = (-1)^k$.

For $k-l = 0$, (5.1) reduces to $e_{k+1,k} = e_{k,k} - e_{k,k-1}$ which is easily checked. For the induction step, the idea is to express every term in (5.1) using (10.3) and then to match up the results using the induction hypothesis. In fact, the left side of (5.1) is

$$\begin{aligned}
(10.4) \quad e_{k+1,l} \quad &= \quad \sum_{j=l+1}^{k+1} \left[-\binom{j}{l}_3 + \binom{j}{l+1}_3\right] e_{k+1,j} \\
&= \quad \sum_{j=l+1}^{k+1} \left[-\binom{j}{l}_3 + \binom{j}{l+1}_3\right] (e_{k,j} - e_{k,j-1} - e_{k-1,j})
\end{aligned}$$

by the induction hypothesis, while the right side is

$$(10.5)\quad e_{k,l} - e_{k,l-1} - e_{k-1,l} = \sum_{j=l+1}^{k} \left[-\binom{j}{l}_3 + \binom{j}{l+1}_3 \right] e_{k,j} - \sum_{j=l}^{k} \left[-\binom{j}{l-1}_3 + \binom{j}{l}_3 \right] e_{k,j} - \sum_{j=l+1}^{k-1} \left[-\binom{j}{l}_3 + \binom{j}{l+1}_3 \right] e_{k-1,j}$$

Of the three sums on the right side of (10.5) the first and the third match with parts of the right side of (10.4); even though the ranges of summation are not quite the same, the offending terms are zero. Thus it remains to show that

$$\sum_{j=l}^{k} \left[-\binom{j}{l-1}_3 + \binom{j}{l}_3 \right] e_{k,j} = \sum_{j=l+1}^{k+1} \left[-\binom{j}{l}_3 + \binom{j}{l+1}_3 \right] e_{k,j-1}$$

If we shift the index of summation on the right, we can rewrite this equation as $S = 0$, where

$$S = \sum_{j=l}^{k} \left[-\binom{j}{l-1}_3 + \binom{j}{l}_3 + \binom{j+1}{l}_3 - \binom{j+1}{l+1}_3 \right] e_{k,j}$$

For $j = l$ the coefficient of $e_{k,j}$ is

$$-\binom{l}{l-1}_3 + \binom{l}{l}_3 + \binom{l+1}{l}_3 - \binom{l+1}{l+1}_3 = -l + 1 + (l+1) - 1 = 1,$$

so that

$$\begin{aligned} S &= e_{k,l} + \sum_{j=l+1}^{k} \left[-\binom{j}{l-1}_3 + \binom{j}{l}_3 + \binom{j+1}{l}_3 - \binom{j+1}{l+1}_3 \right] e_{k,j} \\ &= \sum_{j=l+1}^{k} \left[-\binom{j}{l-1}_3 + \binom{j}{l+1}_3 + \binom{j+1}{l}_3 - \binom{j+1}{l+1}_3 \right] e_{k,j} \end{aligned}$$

by (10.3).
For $j = l + 1$ the new coefficient is

$$-\frac{(l+1)(l+2)}{2} + 1 + \frac{(l+2)(l+3)}{2} - (l+2) = 1,$$

so that

$$\begin{aligned} S &= e_{k,l+1} + \sum_{j=l+2}^{k} \left[-\binom{j}{l-1}_3 + \binom{j}{l+1}_3 + \binom{j+1}{l}_3 - \binom{j+1}{l+1}_3 \right] e_{k,j} \\ &= \sum_{j=l+2}^{k} \left[-\binom{j}{l-1}_3 + \binom{j}{l+2}_3 + \binom{j+1}{l}_3 - \binom{j+1}{l+1}_3 \right] e_{k,j} \end{aligned}$$

by another application of (10.3). At this point all the coefficients are zero by the recursion

$$\binom{j+1}{l}_3 = \binom{j}{l-1}_3 + \binom{j}{l}_3 + \binom{j}{l+1}_3$$

for trinomial coefficients, applied to $\binom{j+1}{l}_3$ and $\binom{j+1}{l+1}_3$. So $S = 0$, completing the proof of (5.1) and hence the proof of Jacobi's identity.

11. Counting potholes with the start(l)ing sequence

Here we generalize the bijection of Theorem 4.3 between odd and even partitions to a bijection of potholes. This allows us to drop the restriction to maximum weight, and we get a parity-reversing bijection between potholes of n with decrement d and those of $n + d + 1$ with decrement $d + 1$. As a result, we will be able to use sequence (4.1) to count potholes of a given decrement.

The bijection results from an operation on shadowed potholes, which we refer to as **sliding**. To slide a shadowed pothole P, modify the statures of its part sizes to arrive at a new pothole P' such that n has high decrement in P' exactly if $n - 1$ has high decrement in P, and so that the same part sizes are bold in P and P'. We can use Table 5 to carry out this operation, several examples of which will soon follow.

In the particular case of a maximum weight partition, as we saw in Section 9 there is a unique pothole of maximum decrement. This pothole has no phantom or negative part, and the only statures are ${}^0_0$, ${}^1_0$, and ${}^2_0$. Under sliding, if i is the smallest missing part, all larger parts that are present have stature ${}^1_0$ and shift up by 1; a new part i^1_0 is introduced. Finally, the parts $1, 2, \ldots, i-1$ remain with their original statures ${}^1_0$ and ${}^2_0$ because each high decrement gets shifted to the next part size (and the **high** decrement of the absent part size 0 shifts to 1); the stature of one of these part sizes is ${}^2_0$ if and only if the part size is bold, and this doesn't get shifted. So in this case, sliding yields exactly the bijection from the proof of Theorem 4.3.

Here are some examples of sliding in general. For the pothole $P = 2^3_0$ of 6, we see that in the shadowed form $2^2_0 2^-$, the part 2 and all nonpositive parts except 2^- have high decrement (while all other parts have low decrement). So in P', the parts 3, 1, and all nonpositive parts except 1^- have high decrement. The part 2 is still the only bold part, so P' has shadowed form $3^1_0 2^1_1 1^1_0 1^-$ and we have $P' = 3^1_0 2^1_1 1^2_1$. Note that P has decrement 0 and that

P' is a pothole of $7 = 6+0+1$ with decrement 1. If we slide the pothole P' again to get P'', parts 4, 2, 1, and all nonpositive parts except 0 will have high decrement, and we find that $P'' = 4_0^1 2_0^2 1_0^1 0^-$, a pothole of $9 = 7+1+1$ with decrement 2. A third sliding yields $P''' = 5_0^1 3_0^1 2_0^2$.

For the pothole $Q = 8_0^1 6_0^1 4_2^1 2_2^2 1_3^0$ of 25 that was shown in Table 6, we can shift the row of 'l's and 'h's in that figure one position to the left (hence the term "sliding") to see that in Q', the parts 9, 7, 2, 1, -2, and all parts ≤ -4 should have high decrement. Since only the parts 2 and 1 are bold, the shadowed form of Q' is $9_0^1 7_0^1 2_0^2 1_0^2 0^- 1^- 3^-$, from which we get $Q' = 9_0^1 7_0^1 3_2^1 2_0^2 1_0^3 0_1^-$.

Theorem 11.1. *If P is a pothole of n with decrement d, and P' is obtained from P by sliding, then P' is a pothole of $n' = n+d+1$ with decrement $d' = d+1$, and P and P' have opposite parity.*

Proof. We begin with the observation that the decrement d of P is equal to the difference between the number m_+ of positive part sizes of high decrement and the number m_- of nonpositive part sizes of low decrement.

Compare P and P' in shadowed form. The decrement of P has increased by 1 in sliding, since we've either lost a nonpositive part size (namely 0) of low decrement or gained a positive part size (namely 1) of high decrement. Therefore, $d' = d+1$; in particular P' is a pothole.

To find the relation between n and n', note that we can account for the changes between P and P' as follows.

(I) For each positive part size i of high decrement in P, starting with the largest such part, change the decrement of part size $i+1$ to high and of part size i to low, contributing $(i+1) - i = 1$ to $n' - n$, regardless of whether i and $i+1$ have normal or bold stature.

(II) If P does not contain a phantom, change the decrement of part size 1 to high, contributing 1 to $n' - n$.

(III) For each negative part i of low decrement in P, change that decrement to high and change the decrement of $i+1$ to low, contributing $i - (i+1) = -1$ to $n' - n$.

If P has no phantom, the contributions of (I), (II), and (III) to $n' - n$ are m_+, 1, and $-m_-$ respectively, while if P has a phantom they are m_+, 0, and $-(m_- - 1)$. In either case, the total contribution is

$$n' - n = m_+ - m_- + 1 = d + 1,$$

as desired. Finally, we consider parity. By the comment at the end of Section 7 and the first observation of this proof, the parity of a pothole is given by the number of bold statures plus the decrement. Because the bold part sizes are unchanged and $d' = d+1$, the parities of P and P' are opposite. □

Evidently, any pothole of positive decrement can be obtained by sliding, and the operation is invertible. Therefore sliding provides a bijection between even, respectively odd, potholes of n with decrement d and odd, respectively even, potholes of $n+d+1$ with decrement $d+1$, or in the other direction, for $d>0$, between potholes of n with decrement d and potholes of the opposite parity of $n-d$ with decrement $d-1$. Here $n-d$ will always be nonnegative, and will be positive except in the case of the lone pothole 1_0^1 of $n=1$ with decrement $d=1$, which corresponds to the empty pothole of 0.

Carrying out this "reverse sliding" repeatedly, we can match any pothole of n with decrement d to a pothole of $n-\frac{1}{2}d(d+1)$ with decrement 0, where $n-\frac{1}{2}d(d+1)$ will be 0 exactly if we started with the exceptional pothole of a triangular number. Thus to be able to count potholes with a given decrement, it is enough to be able to count potholes of any positive integer with decrement 0. This also enables us to count potholes of specified decrement and parity, as sliding (in either direction) reverses parity.

To count potholes of m with decrement 0, we use repeated sliding (forward). After sliding m times, potholes of m with decrement 0 will yield potholes of $m+(1+2+\ldots+m)=\frac{1}{2}m(m+1)+m$ with decrement m. As we saw at the beginning of Section 9, these last potholes are in one to one correspondence with partitions of $\frac{1}{2}m(m+1)+m$ of maximum weight. Now recall from Section 4 that $u^{\pm}(m)$, the m-th term of sequence (4.1), counts partitions of $\frac{1}{2}k(k+1)+m$ of maximum weight and parity given by the $\pm$ sign for $0\le m\le k$. So taking $k=m$ we see that $u^{\pm}(m)$ counts potholes of m with decrement 0, and the parity of the $\pm$ sign, i.e. the parity of m if the sign is $+$ and its opposite if the sign is $-$.

In summary, we have the following theorem.

Theorem 11.2. *The numbers of potholes of n with decrement d are given by*

$$\begin{aligned} p^{+}_{n,d} &= u^{\pm}\left(n-\frac{1}{2}d(d+1)\right) \\ p^{-}_{n,d} &= u^{\mp}\left(n-\frac{1}{2}d(d+1)\right) \end{aligned} \tag{11.1}$$

where $\pm$ is the sign of $(-1)^d$.

For example, we saw earlier that for the pothole $P=2_0^3$ of 6, sliding three times yields $P'''=5_0^1 3_0^1 2_0^2$. Sliding three more times, we arrive at the pothole $8_0^1 6_0^1 5_0^1 3_0^1 2_0^2 1_0^1$ of $27=\frac{1}{2}6(6+1)+6$ with decrement 6, corresponding to the maximum weight partition $8\ 6\ 5\ 3\ 2^2\ 1$. There are $u^{\pm}(6)=20$ such partitions of 27 of each parity, and they correspond to the 20 potholes of 6 with decrement 0 enumerated in the left columns of Table 8. Similarly, the complete set of 24 potholes of 4 shown in Section 6 consists of $u^{\pm}(4)=7$ of each parity with decrement 0, $u^{\pm}(3)=4$ of each parity with decrement 1, and $u^{\pm}(1)=1$ of each parity with decrement 2.

12. Relation to a Jacobi theta function

We now prove Theorem 4.1, that

$$\sum_{m=0}^{\infty}[u^+(m)+u^-(m)]q^m = \frac{1}{\vartheta_4(q)}$$

$$= \frac{1}{(1-q)^2(1-q^2)(1-q^3)^2(1-q^4)(1-q^5)^2\cdots}$$

$$= \prod_{n=1}^{\infty}\frac{1}{(1-q^n)(1-q^{2n-1})}$$

This is the generating function for the number of divided partitions of m, where the right division contains only odd parts. The latter partitions are equinumerous with partitions into distinct parts [1, p. 5, Corollary 1.2] so we'll use those instead. For example, the relevant divided partitions of 0, 1, 2, 3 and 4 are given in Table 10.

Note that by Theorem 11.2, $u^+(m)+u^-(m) = p^+_{m,0}+p^-_{m,0}$. Thus we must now show that the number of such divided partitions of m is equal to the number of potholes of m of decrement 0.

It is straightforward to check that, for positive i, an unshadowed stature $i^{m_i}_{h_i}$ is reflected in shadowed form by exactly m_i of the following three features: (i) high decrement at part size i; (ii) a shadow, of low decrement, at part size $-i$; and (iii) a bold stature at part size i. The phantom is represented by low decrement at part size 0. A shadowed pothole is completely determined by which of these features it possesses for each $i \geq 0$. So we can associate to it a divided partition of m, with three divisions, each into distinct parts: the first contains all positive part sizes i with high decrement, the second all *non-negative* part sizes i such that $-i$ has low decrement, and the third all positive part sizes i of bold stature. Since $d = 0$, it follows from the observation on decrement that starts the proof of Theorem 11.1 that we get only divided partitions whose first two components have an equal

m	divided partitions; right division having distinct parts	number
0	$\{\,\vert\,\}$	1
1	$\{1\vert\,\}$ $\{\,\vert 1\}$	2
2	$\{2\vert\,\}$ $\{1^2\vert\,\}$ $\{1\vert 1\}$ $\{\,\vert 2\}$	4
3	$\{3\vert\,\}$ $\{21\vert\,\}$ $\{1^3\vert\,\}$ $\{2\vert 1\}$ $\{1^2\vert 1\}$ $\{1\vert 2\}$ $\{\,\vert 3\}$ $\{\,\vert 21\}$	8
4	$\{4\vert\}$ $\{31\vert\}$ $\{2^2\vert\}$ $\{21^2\vert\}$ $\{1^4\vert\}$ $\{3\vert 1\}$ $\{21\vert 1\}$	
	$\{1^3\vert 1\}$ $\{2\vert 2\}$ $\{1^2\vert 2\}$ $\{1\vert 3\}$ $\{1\vert 21\}$ $\{\vert 4\}$ $\{\vert 31\}$	14

Table 10.

number of parts. Conversely, for any such tripartite divided partition we have a unique pothole of decrement 0.

We now have two kinds of divided partition, one with two divisions and one with three, and we want to establish that they are equal in number. The final components of each of these kinds of divided partition are the same, so we'll be done if we show that the number of (unrestricted) partitions P of m is equal to the number of divided partitions $(P_1|P_2)$ of m, such that both P_1 and P_2 have distinct parts; they have equally many parts; and 0 is allowed to be a part of P_2, but not of P_1.

We establish this by a simple manipulation of Ferrers diagrams. Split the Ferrers diagram of P by a diagonal line passing just below its main diagonal; then the rows of the upper section give P_1, and the columns of the lower section give P_2. This is readily seen to be bijective.

For example, Figure 2 illustrates the mapping between $P = 5\ 4^2\ 2\ 1^4$ and $(P_1|P_2) = (5\ 3\ 2|7\ 2\ 0)$.

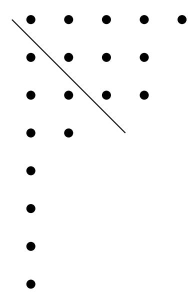

FIGURE 2. The mapping between $P = 5\ 4^2\ 2\ 1^4$ and $(P_1|P_2) = (5\ 3\ 2|7\ 2\ 0)$.

To prove Theorem 4.2, that $u^+(m)$ is odd just if m is square, start with the power series

$$\vartheta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - + - \cdots$$

Define $g(n)$ to be $u^+(n) + u^-(n)$, so that $g(0) = 1$ and $g(n)$ is even for $n > 0$, and multiply by

$$\frac{1}{\vartheta_4(q)} = \sum_{n=0}^{\infty}(u^+(n) + u^-(n))q^n = \sum_{n=0}^{\infty} g(n)q^n$$

to get the recurrence relation

$$g(n) - 2g(n-1) + 2g(n-4) - 2g(n-9) + 2g(n-16) - + - \cdots = 0$$

or equivalently

$$g(n) = 2g(n-1) - 2g(n-4) + 2g(n-9) - 2g(n-16) - + - \cdots$$

for $n > 0$. If n is nonsquare, each $g(i)$ on the right side of this recurrence is even, so 4 divides $g(n)$ and $u^+(n) = \frac{1}{2}g(n)$ is even. But if n is square, a term $\pm 2g(0) = \pm 2$ appears, and $g(n) \equiv 2 \pmod 4$, so $u^+(n)$ is odd.

13. Other combinatorial connections

Consider the falling diagonals of Table 3. The j-th diagonal has entries $e_{k,k-j}$. The signs of the entries in the first few diagonals suggest looking at $f_{k,j} = (-1)^{k-j}e_{k,k-j}$, which equals 1 for $j = 0$ and $k - 1$ for $j = 1$. If we multiply our recurrence relation (5.1) from Section 5 by $(-1)^l$ and then set $l = k + 1 - j$, we can rewrite the relation as

$$f_{k+1,j} - f_{k,j} = f_{k,j-1} - f_{k-1,j-2}$$

Note that the left-hand side of this recurrence is the forward difference of $f_{k,j}$ for fixed j. It follows by induction on j that for each j, the values of $f_{k,j}$ are given by a polynomial of degree j in k. To avoid denominators, we scale these polynomials by multiplying by $j!$. This yields the polynomials $E_j(k)$ shown in Table 11, for which $E_j(k) = (-1)^{k-j}j!e_{k,k-j}$ when k and j are nonnegative integers with $j \le k$.

If we evaluate these polynomials for negative values of k, or, equivalently, assume that the recurrence (5.1) holds for negative values of l, then we may define $e_{k,l}$ for all integers k, l. Some of these values are exhibited in Table 12.

The leading linear factors in Table 11 correspond to the zero values for $-k \le l < 0$ and $l = 0$ or 2, $k = 3g + 1$ evident in Table 12.

For $l = -1$, we have $e_{k,l} = 0$ except for $e_{0,-1} = 1$ and $e_{-1,-1} = -1$. The recurrence (5.1) may be written

$$(-1)^k e_{k,l-1} = (-1)^{k-1}e_{k-1,l} + (-1)^k e_{k,l} + (-1)^{k+1}e_{k+1,l}$$

so that for $l < 0$ we have, with a reversal of sign, symmetry about the line $k = -\frac{1}{2}$:

$$e_{k,l} = -e_{-1-k,l} = (-1)^k \left[\binom{-l-1}{k}_3 + \binom{-l-1}{k+1}_3\right] \tag{13.1}$$

where the second equality follows from the previous equation by induction on $-l$, since trinomial coefficients satisfy the recurrence

$$\binom{-l+1}{k}_3 = \binom{-l}{k-1}_3 + \binom{-l}{k}_3 + \binom{-l}{k+1}_3. \tag{13.2}$$

The identity (13.1) can be made to hold even for $l \ge 0$ by extending the definition of the trinomial coefficients recursively to negative integers so that (13.2) holds everywhere and $\binom{-l}{k}_3 = 0$ for $l > k$.

The coefficients of the polynomials $E_j(k)$ form the array shown in Table 13 (see page 109).

j	$E_j(k) = (-1)^{k-j} j! e_{k,k-j}$
0	1
1	$k-1$
2	$(k-1)(k-4)$
3	$(k-2)(k^2-10k+15)$
4	$(k-2)(k-3)(k-4)(k-13)$
5	$(k-3)(k-4)(k-7)(k^2-21k+50)$
6	$(k-3)(k-4)(k-5)(k^3-39k^2+410k-1152)$
7	$(k-4)(k-5)(k-6)(k-7)(k^3-48k^2+599k-1602)$
8	$(k-4)(k-5)(k-6)(k-7)(k-10)(k^3-60k^2+971k-3600)$
9	$(k-5)(k-6)(k-7)(k-8)(k^5-91k^4+2861k^3-38669k^2+225330k-458568)$
10	$(k-5)(k-6)(k-7)(k-8)(k-9)(k-10)(k^4-100k^3+3275k^2-40280k+145104)$
11	$(k-6)(k-7)(k-8)(k-9)(k-10)(k-13)(k^5-123k^4+5251k^3-94707k^2+700078k-1756920)$

TABLE 11. The polynomials $E_j(k)$.

$l =$		–10	–9	–8	–7	–6	–5	–4	–3	–2	–1	0	1	2	3
n	k														
10	–5	–1296	–378	–105	–27	–6	–1	0	0	0	0	0	0	0	0
6	–4	2436	770	238	71	20	5	1	0	0	0	0	0	0	0
3	–3	–3858	–1288	–427	–140	–45	–14	–4	–1	0	0	0	0	0	0
1	–2	5211	1800	623	216	75	26	9	3	1	0	0	0	0	0
0	–1	–6046	–2123	–750	–267	–96	–35	–13	–5	–2	–1	0	0	0	0
0	0	6046	2123	750	267	96	35	13	5	2	1	1	0	0	0
1	1	–5211	–1800	–623	–216	–75	–26	–9	–3	–1	0	0	–1	0	0
3	2	3858	1288	427	140	45	14	4	1	0	0	–1	–1	1	0
6	3	–2436	–770	–238	–71	–20	–5	–1	0	0	0	–1	1	2	–1
10	4	1296	378	105	27	6	1	0	0	0	0	0	3	0	–3
15	5	–570	–148	–35	–7	–1	0	0	0	0	0	1	2	–5	–2
21	6	201	44	8	1	0	0	0	0	0	0	1	–2	–7	6
28	7	–54	–9	–1	0	0	0	0	0	0	0	0	–5	0	15
36	8	10	1	0	0	0	0	0	0	0	0	–1	–3	12	9
45	9	–1	0	0	0	0	0	0	0	0	0	–1	3	15	–18
55	10	0	0	0	0	0	0	0	0	0	0	0	7	0	–42

TABLE 12. Extension of Table 3 showing $e_{k,l}$ for negative k and l.

These coefficients are of interest in their own right. We may recast (5.1) in terms of the $E_j(k)$ using the definition $E_j(k) = (-1)^{k-j} j! e_{k,k-j}$. After routine simplifications, and writing $j = k - l$, we get

$$E_{j+1}(k+1) - E_{j+1}(k) = (j+1)E_j(k) - j(j+1)E_{j-1}(k-1).$$

Now let $F_c(j)$ denote the coefficient of k^{j-c} in $E_j(k)$, so that the functions F_c give leftward diagonals of Table 13. To get a handle on these F_c we equate coefficients of k^{j-c} in the last display, using the binomial theorem to expand the first and last Es:

$$\sum_{i=0}^{c+1}\left(F_i(j+1)\binom{j+1-i}{c+1-i}\right) - F_{c+1}(j+1)$$
$$= (j+1)F_c(j) - j(j+1)\sum_{i=0}^{c-1}\left(F_i(j-1)(-1)^{-1-i+c}\binom{j-1-i}{c-1-i}\right).$$

The terms involving F_{c+1} cancel, and the largest subscript occurring among the remaining F_i is $i = c$. We isolate the terms containing F_c to get

$$(13.3)\quad (j+1-c)F_c(j+1) - (j+1)F_c(j) = -\sum_{i=0}^{c-1}\left(F_i(j+1)\binom{j+1-i}{c+1-i}\right) - j(j+1)\sum_{i=0}^{c-1}\left(F_i(j-1)(-1)^{-1-i+c}\binom{j-1-i}{c-1-i}\right).$$

From here we deduce by induction that $F_c(j)$ is a polynomial in j of degree $2c$ divisible by $j(j-1)\cdots(j-(c-1))$. For $c = 0$ the right side of (13.3) vanishes. For positive c we divide (13.3) through by $(j+1)j\cdots(j-(c-1))$, so that the left side becomes the forward difference

$$\frac{F_c(j+1)}{(j+1)((j+1)-1)\cdots((j+1)-(c-1))} - \frac{F_c(j)}{j(j-1)\dots(j-(c-1))}$$

and the right side by hypothesis is a polynomial of degree $c-1$. Then $F_c(j)/\prod_{k=0}^{c-1}(j-k)$ is a polynomial in j of degree c and the induction carries through.

For $0 \le i \le 5$ the polynomials $F_i(j)$ are shown in Table 14. The values for $i = 1$ are the negatives of the pentagonal numbers of positive rank.

The constant term of $E_j(k)$ is $(-1)^j j! a_j$ where, from (13.1),

$$a_j = e_{0,-j} = \binom{j-1}{0}_3 + \binom{j-1}{1}_3 = \binom{j}{0}_3 - \binom{j-1}{1}_3 = \frac{1}{2}\left[\binom{j}{0}_3 + \binom{j-1}{0}_3\right],$$

1										
1	−1									
1	−5	4								
1	−12	35	−30							
1	−22	143	−362	312						
1	−35	405	−2065	4814	−4200					
1	−51	925	−7965	35434	−78744	69120				
1	−70	1834	−24010	173929	−709240	1525236	−1345680			
1	−92	3290	−61040	655529	−4235588	16255420	−34148400	30240000		
1	−117	5478	−136962	2047521	−19225773	114508232	−420266268	867387888	−770394240	
1	−145	8610	−279330	5554353	−71516865	609204140	−3413513420	12105991296	−24643333440	21939724800

TABLE 13. Coefficients of the polynomials $E_j(k)$.

i	$F_i(j)$, the coefficient of k^{j-i} in $E_j(k)$
0	1
1	$-j(3j-1)/2$
2	$j(j-1)(27j^2-43j+26)/24$
3	$-j(j-1)(j-2)(27j^3-102j^2+199j-168)/48$
4	$j(j-1)(j-2)(j-3)(1215j^4-8370j^3+31925j^2-68898j+64312)/5760$
5	$-j(j-1)(j-2)(j-3)(j-4)(2187j^5-23895j^4+147195j^3-565377j^2+1256322j-1236960)/34560$

TABLE 14. The polynomials $F_i(j)$.

the mean of two consecutive central trinomial coefficients, whose generating function is $(1-2x-3x^2)^{-\frac{1}{2}}$ (see [18, vol. 2, 6.3.8]), so that the generating function for a_j is

$$\frac{1}{2}\left[1+(1+x)(1-2x-3x^2)^{-\frac{1}{2}}\right]=\frac{1}{2}\left[\sqrt{\frac{1+x}{1-3x}}+1\right].$$

The sequence $\{a_j\}$ is A005773 in [17]; it is the number of directed animals of size j [3, 6, 8, 21] [4, p.226, Exercise 15]; the number of directed j-ominoes in standard position [7, p. 338–343]; the number of ordered trees with j edges whose nodes other than the root node have outvalence at most two; the number of symmetric Dyck paths of semilength $2j-1$ (or of semilength $2j$) with no peaks at an even level [13]; the number of paths of $j-1$ steps U=(1,1), H=(1,0), or D=(1,−1) starting at (0,0) and not going below $y=0$ (i.e., left factors of length $j-1$ of Motzkin paths) [2]; and the number of UDU-free paths of j U-steps and j D-steps, starting with U. Finally, a_j is the number of base 3 j-digit numbers with digital sum j [15].

See also [18, Vol. 2, pp. 242–243, Ex. 6.46].

We can also deduce formulas for the first few columns of Table 3. These are shown in Table 15 where we have written $k=6g+h$ with $0\le h\le 5$.

Note the similarity of rows $h=0$ and 5, of 1 and 4, and of 2 and 3. The values of $e_{k,2}$ are signed pentagonal numbers.

Denote the sum $\sum_{l=0}^{\infty} e_{k,l}r^l$ by $S_k(r)$. It is a polynomial in r of degree k, since $e_{k,l}=0$ if $l>k$. If we multiply the recurrence of Theorem 5.1, namely

$$e_{k+1,l}=e_{k,l}-e_{k,l-1}-e_{k-1,l}$$

by r^l and sum over l, we obtain the following theorem.

Theorem 13.1.

$$S_{k+1}(r)-(1-r)S_k(r)+S_{k-1}(r)=0 \qquad (k\neq 0). \tag{13.4}$$

In Table 16 we list values of the polynomials $S_k(r)$, so that the columns are sequences which will each satisfy the recurrence (13.4).

All of the entries in Table 16 are values of Chebyshev polynomials of the second kind of even degree. In fact, $U_n(\cos\phi)=\sin(n+1)\phi\csc\phi$ [4, p.50, formula 14k], so that

$$\begin{aligned} U_{2k+2}(\cos\phi)+U_{2k-2}(\cos\phi) &= \frac{\sin(2k+3)\phi+\sin(2k-1)\phi}{\sin\phi} \\ &= \frac{2\sin(2k+1)\phi\cos 2\phi}{\sin\phi} \\ &= (4\cos^2\phi-2)U_{2k}(\cos\phi) \end{aligned}$$

a recurrence which coincides with that of Theorem 13.1 with $r=4\cos^2\phi-1$ and

$$S_k(r)=(-1)^kU_{2k}\left(\frac{\sqrt{r+1}}{2}\right).$$

h	$e_{k,0}$	$e_{k,1}$	$e_{k,2}$	$e_{k,3}$	$e_{k,4}$
0	1	$-2g$	$-g(6g+1)$	$2g^2(2g+1)$	$\frac{1}{6}g(2g-1)(2g+1)(9g+1)$
1	0	$-(4g+1)$	0	$g(2g+1)(4g+1)$	$-\frac{1}{3}g(2g+1)(4g+1)$
2	-1	$-(2g+1)$	$(2g+1)(3g+1)$	$g(2g+1)^2$	$-\frac{1}{6}g(g+1)(2g+1)(18g+7)$
3	-1	$2g+1$	$(2g+1)(3g+2)$	$-(g+1)(2g+1)^2$	$-\frac{1}{6}g(g+1)(2g+1)(18g+11)$
4	0	$4g+3$	0	$-(g+1)(2g+1)(4g+3)$	$\frac{1}{3}(g+1)(2g+1)(4g+3)$
5	1	$2g+2$	$-(g+1)(6g+5)$	$-2(g+1)^2(2g+1)$	$\frac{1}{6}(g+1)(2g+1)(2g+3)(9g+8)$

TABLE 15. Formulas for the first few columns of Table 3.

$r =$	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8
$k = 0$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	6	5	4	3	2	1	0	–1	–2	–3	–4	–5	–6	–7	–8
2	41	29	19	11	5	1	–1	–1	1	5	11	19	29	41	55
3	281	169	91	41	13	1	–1	1	1	–7	–29	–71	–139	–239	–377
4	1926	985	436	153	34	1	0	1	–2	9	76	265	666	1393	2584
5	13201	5741	2089	571	89	1	1	–1	1	–11	–199	–989	–3191	–8119	...
6	90481	33461	10009	2131	233	1	1	–1	1	13	521	3691	15289	47321	
7	...	...	47956	7953	610	1	0	1	–2	–15	–1364	–13775	–73254	...	
8			...	29681	1597	1	–1	1	1	17	3571	51409	...		
9				...	4181	1	–1	–1	1	–19	–9349	...			
10					10946	1	0	–1	–2	21	24476				
11					28657	1	1	1	1	–23	–64079				
12					75025	1	1	1	1	25	...				
13					...	1	0	–1	–2	–27					
14						1	–1	–1	1	29					

TABLE 16. Values of $S_k(r) = \sum e_{k,l} r^l$ with the rows of Table 3 as coefficients.

r	OEIS #	r	OEIS #	r	OEIS #	r	OEIS #	r	OEIS #
−14	...	−8	A070998	−2	A001519	4	A002878	10	A057081
−13	A001570	−7	A070997	−1	A000012	5	A001834	11	A054320
−12	A085260	−6	A049685	0	A011655	6	A030221	12	A097783
−11	A077417	−5	A001653	1	A057077	7	A002315	13	A077416
−10	A078922	−4	A004253	2	A057079	8	A033890	14	A126866
−9	A072256	−3	A001835	3	A005408	9	A057080	15	A028230

TABLE 17. List of columns with their OEIS numbers.

The initial values agree, namely 1 for $k = 0$ and $-r$ for $k = 1$.

Explicitly, unless $r = -1$ or 3 (for which $\alpha = \beta$), $S_k(r) = A\alpha^k + B\beta^k$ where

$$A, B = \left\{ 1 \mp \frac{r+1}{\sqrt{(r+1)(r-3)}} \right\} / \, 2$$

and

$$\alpha, \beta = \left\{ 1 - r \pm \sqrt{(r+1)(r-3)} \right\} / \, 2.$$

The $S_k(r)$ are also solutions of the Brahmagupta-Bhaskara-Pell equation

$$(r-3)x^2 - (r+1)y^2 = -4$$

where $x = S_k(r)$, $y = S_k(2-r)$ and we may interchange r and $2-r$. The generating function for column r is $(1-x)/\left(1-(1-r)x+x^2\right)$.

Many of the columns are sequences with other properties. Table 17 is a list of their OEIS [17] numbers, where additional references can be found. Finally, here are some additional properties, often underlining the $(r, 2-r)$ duality.

Columns $r = -13$, 11 *and* 15*:* these are numbers whose square is respectively a hex number [5, p. 41] (difference of consecutive cubes [19]); a star number (centred dodecagonal number); and an octagonal number [12, pp. 102–104].

Columns $r = -2$, 4, −6 *and* 8*:* these columns are respectively F_{2k+1}; $(-1)^k L_{2k+1}$; $\frac{1}{3}L_{4k+2}$; and $(-1)^k F_{4k+2}$, where F and L are Fibonacci and Lucas numbers.

Columns $r = -5$, −4, −3 *and* −2*:* these are numbers of domino tilings (perfect matchings) of the respective $-(r+1)(2k-1)$-edge graphs $P_{2k} \times K_{1,4}$, $P_{2k} \times C_3$, $P_{2k} \times P_3$ and $P_{2k} \times P_2$, where P_n is a path with n vertices, $K_{1,4}$ is the bipartite graph joining 1 vertex to 4 others, and C_3 is a 3-circuit.

Column $r = 7$*:* this column contains NSW numbers [14].

Column $r = 3$*:* This column is, of course, $(-1)^k(2k+1)$, the right side of (1.1), which is where we came in!

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