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MARTIN'S AXIOM AND ALMOST DISJOINT FAMILIES

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Abstract. Assuming Martin's Axiom and $\aleph_1 < 2^{\aleph_0}$, we show that, for any $\kappa, \lambda < 2^{\aleph_0}$ and any almost disjoint family $\{a_i : i < \lambda\}$ of countable subsets of κ, there is a partition $\{p_n : n \in \omega\}$ of κ so that $p_n \cap a_i$ is finite for each $\langle i, n\rangle \in \lambda \times \omega$.

1. Introduction

Almost disjoint families are frequently used in set theory, general topology [6, Ch. 3, §11] and Boolean algebras [1, 5]. We recall that if D is a set and $\mathcal{A} \subseteq \mathcal{P}(D)$, the set $\mathcal{A}$ is an almost disjoint family, if $|a| = \aleph_0$ for any $a \in \mathcal{A}$, and $a \cap b$ is finite for every distinct members a and b of $\mathcal{A}$. As a consequence of [3, Theorem 2.3] there is an almost disjoint family $\{a_i : i < \aleph_1\}$ of countable subsets of $\aleph_1$ such that for every uncountable subset b of $\aleph_1$, $a_i \subseteq b$ for some $i < \aleph_1$. This result is used to construct various "increasing chains" of superatomic Boolean algebras that have well-founded generating sublattices whose union is a superatomic Boolean algebra and is not generated by a well-founded sublattice. On the other hand, M. Rubin shows, under Martin's Axiom, Theorem 1.1 below for $\kappa = \aleph_1$ (see [3, Proposition 2.5]). In this present work, we give a complete and detailed proof of Rubin's result for any $\kappa < 2^{\aleph_0}$. In Remark 2 we will see that Theorem 1.1 cannot be extended in cardinality $2^{\aleph_0}$.

Theorem 1.1. *Assume Martin's Axiom and $\aleph_1 < 2^{\aleph_0}$. Let $\kappa, \lambda < 2^{\aleph_0}$ and $\mathcal{A} := \{a_i : i < \lambda\}$ be an almost disjoint family of countable subsets of κ. Then there is a partition $\{p_n : n \in \omega\}$ of κ such that for every $n \in \omega$, $p_n \cap a$ is finite for any $a \in \mathcal{A}$.*

Proof. Let $\langle \mathbb{P}, \leq\rangle$ be the poset defined as follows. A member of $\mathbb{P}$ has the form $\langle \sigma, \eta\rangle$, where

(1) σ is a finite subset of $\omega \times \kappa$, and for every $\alpha \in \kappa$ there is at most one $n \in \omega$ such that $\langle n, \alpha\rangle \in \sigma$, and
(2) η is a finite subset of λ.

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For $\langle \sigma_1, \eta_1 \rangle, \langle \sigma_2, \eta_2 \rangle \in \mathbb{P}$, we define $\langle \sigma_1, \eta_1 \rangle \leq \langle \sigma_2, \eta_2 \rangle$ if $\sigma_1 \subseteq \sigma_2$, $\eta_1 \subseteq \eta_2$, and for every $i \in \eta_1$ and $n \in \mathrm{Dom}(\sigma_1)$, $(\{n\} \times a_i) \cap \sigma_1 = (\{n\} \times a_i) \cap \sigma_2$.

For $\langle \sigma, \eta \rangle \in \mathbb{P}$, the interpretation of "$\langle n, \beta \rangle \in \sigma$ and $i \in \eta$" will be $\beta \in p_n \cap a_i$.

The proof that $\mathbb{P}$ satisfies c.c.c. (Claim 1.4) uses a non-standard argument that we can find in [2]. First, we show that some sets are dense.

Claim 1.2. *For every $\alpha < \kappa$, $D_\alpha := \{\langle \sigma, \eta \rangle \in \mathbb{P} : \alpha \in \mathrm{Rng}(\sigma)\}$ is dense in $\mathbb{P}$.*

Proof. Let $\langle \sigma, \eta \rangle \in \mathbb{P}$ and $\alpha \in \kappa$. If $\alpha \in \mathrm{Rng}(\sigma)$ there is nothing to prove. Suppose that $\alpha \notin \mathrm{Rng}(\sigma)$. Let $\ell \in \omega \backslash \mathrm{Dom}(\sigma)$, $\sigma' := \sigma \cup \{\langle \ell, \alpha \rangle\}$, and $\eta' = \eta$. Then $\langle \sigma', \eta' \rangle \in \mathbb{P}$ and $\langle \sigma, \eta \rangle \leq \langle \sigma', \eta' \rangle$. This is so, because if $i \in \eta$ and $n \in \mathrm{Dom}(\sigma)$, then $\langle \ell, \alpha \rangle \notin \{n\} \times a_i$. □

Claim 1.3. *For every $i < \lambda$, $E_i := \{\langle \sigma, \eta \rangle \in \mathbb{P} : i \in \eta\}$ is dense in $\mathbb{P}$.*

Proof. Let $\langle \sigma, \eta \rangle \in \mathbb{P}$ and $i \in \lambda$. If $i \in \eta$ there is nothing to prove. If $i \notin \eta$, then $\langle \sigma, \eta \rangle \leq \langle \sigma, \eta \cup \{i\} \rangle \in E_i$. □

Claim 1.4. *$\langle \mathbb{P}, \leq \rangle$ is c.c.c.*

Proof. By way of contradiction, let $\{\langle \sigma_\mu, \eta_\mu \rangle : \mu \in \aleph_1\}$ be a set of pairwise incompatible conditions. Let $F = \{\eta_\mu : \mu \in \aleph_1\}$. Suppose first that F is countable. In that case, there are $A \subseteq \aleph_1$ and $\eta \in F$ such that $|A| = \aleph_1$ and $\eta_\mu = \eta$ for every $\mu \in A$. Next suppose $|F| = \aleph_1$. There are $m \in \omega$ and an uncountable subset A' of $\aleph_1$ such that $|\eta_\mu| = m$ for every $\mu \in A'$. By the Δ-lemma, there are an uncountable subset A of A' and a finite set η such that $\eta_\mu \cap \eta_\nu = \eta$ for distinct $\mu, \nu \in A$. In the two cases, $|A| = \aleph_1$, and thus we may assume that $A = \aleph_1$. That is, either

(1) for every $\mu \in \aleph_1$, $\eta_\mu = \eta$, or
(2) for every distinct $\mu, \nu \in \aleph_1$, we have $\eta_\mu \neq \eta_\nu$, $\eta_\mu \cap \eta_\nu = \eta$, and $|\eta_\mu| = |\eta_\nu|$.

Since for every $\mu \in \aleph_1$, $\mathrm{Dom}(\sigma_\mu)$ is a finite subset of ω, we may also suppose that there is a finite set δ of ω, such that

(3) for every $\mu \in \aleph_1$, $\mathrm{Dom}(\sigma_\mu) := \delta$.

Let $\rho_\mu = \mathrm{Rng}(\sigma_\mu)$ for $\mu < \aleph_1$. So ρ_μ is a finite subset of κ. Similiar to the case of η above, we may also assume that there is a finite subset ρ of κ such that either

(4) for every $\mu \in \aleph_1$, $\rho_\mu = \rho$, or
(5) for every distinct $\mu, \nu \in \aleph_1$, we have $\rho_\mu \neq \rho_\nu$, $\rho_\mu \cap \rho_\nu = \rho$ and $|\rho_\mu| = |\rho_\nu|$.

Since $\delta \times \rho$ is finite, there is an uncountable subset B of $\aleph_1$ such that for every distinct $\mu, \nu \in B$, $\sigma_\mu \cap (\delta \times \rho) = \sigma_\nu \cap (\delta \times \rho)$. Hence, we can suppose also that for any $\mu, \nu \in \aleph_1$, $k \in \delta$ and $\alpha \in \rho$,

(6) $\langle k, \alpha \rangle \in \sigma_\mu$ iff $\langle k, \alpha \rangle \in \sigma_\nu$.

Now, we consider $\langle\langle\sigma_\mu,\eta_\mu\rangle : \mu\in\aleph_1\rangle$ as a sequence of pairwise incompatible conditions. Let $\mu\in\aleph_1$. We define an ordinal $\beta_\mu\in\aleph_1$ and a countable subset S_μ of κ satisfying the following properties:

(7) $\bigcup\{a_i : i\in\eta_\mu\}\subseteq S_\mu$, $\beta_\mu\geq\mu$ and $\rho_\nu\cap S_\mu=\rho$ for every $\nu>\beta_\mu$.

Let $T=\bigcup\{a_i : i\in\eta_\mu\}$ and $\Lambda=\{\zeta\in\aleph_1 : (\rho_\zeta\backslash\rho)\cap T\neq\varnothing\}$. Let $S_\mu=T\cup\bigcup\{\rho_\zeta : \zeta\in\Lambda\}$ and $\beta'_\mu=\sup(\Lambda)$. (Notice that $|T|=\aleph_0$ and thus S_μ is countable.) Assume that $(\rho_\nu\backslash\rho)\cap S_\mu\neq\varnothing$. First, if $(\rho_\nu\backslash\rho)\cap T\neq\varnothing$, then $\nu\in\Lambda$, and thus $\nu\leq\beta'_\mu$. Next, suppose that $(\rho_\nu\backslash\rho)\cap\left(\bigcup\{\rho_\zeta : \zeta\in\Lambda\}\right)\neq\varnothing$. Then there is $\zeta\in\Lambda$ such that $(\rho_\nu\backslash\rho)\cap\rho_\zeta\neq\varnothing$. That is $(\rho_\nu\backslash\rho)\cap(\rho_\zeta\backslash\rho)\neq\varnothing$, and thus $\nu=\zeta\in\Lambda$. Hence $\nu\leq\beta'_\mu$. So S_μ and $\beta_\mu:=\max(\beta'_\mu,\mu)$ are as required in (7). Note that in (4), $\Lambda=\varnothing$ and thus $S_\mu=T$ and $\beta_\mu=\mu$ (because $\beta'_\mu:=\sup(\Lambda)=0$).

By (7), using induction, and renaming a subsequence, we may assume that there is a family $\langle S_\mu : \mu\in\aleph_1\rangle$ of countable subsets of κ such that $\langle\langle\sigma_\mu,\eta_\mu\rangle : \mu\in\aleph_1\rangle$ satisfies the following properties:

(8) if $i\in\eta_\mu$ then $a_i\subseteq S_\mu$, and

(9) if $\nu>\mu$, then $\rho_\nu\cap S_\mu=\rho$.

For $\mu<\nu<\omega_1$, let $r_\mu:=\langle\sigma_\mu,\eta_\mu\rangle$ and $r_{\mu,\nu}:=\langle\sigma_\mu\cup\sigma_\nu,\eta_\mu\cup\eta_\nu\rangle$. We show that the following condition holds:

(10) for every $\mu<\nu<\omega_1$, $r_{\mu,\nu}\in\mathbb{P}$ and $r_\mu\leq r_{\mu,\nu}$.

We prove first that $r_{\mu,\nu}\in\mathbb{P}$. Let $\alpha\in\kappa$ and distinct m,n such that $\langle m,\alpha\rangle,\langle n,\alpha\rangle\in\sigma_\mu\cup\sigma_\nu$. For instance $\langle m,\alpha\rangle\in\sigma_\mu$ and $\langle n,\alpha\rangle\in\sigma_\nu$. By (6), $\alpha\notin\rho$, and thus $\alpha\in\rho_\mu\backslash\rho$ and $\alpha\in\rho_\nu\backslash\rho$, which contradicts (5). Now, it is trivial to see that $r_{\mu,\nu}\in\mathbb{P}$. Next we show that $r_\mu\leq r_{\mu,\nu}$. Trivially, $\sigma_\mu\subseteq\sigma_\mu\cup\sigma_\nu$ and $\eta_\mu\subseteq\eta_\mu\cup\eta_\nu$. Let $i\in\eta_\mu$ and $n\in\delta:=\mathrm{Dom}\,(\sigma_\mu)$. Let $\alpha\in a_i$ be such that $\langle n,\alpha\rangle\in\sigma_\nu$. It suffices to prove that $\langle n,\alpha\rangle\in\sigma_\mu$. By way of contradiction, suppose that $\langle n,\alpha\rangle\notin\sigma_\mu$. By (6), $\langle n,\alpha\rangle\notin\delta\times\rho$. Hence $\alpha\notin\rho$ and thus $\alpha\in\rho_\nu\backslash\rho\subseteq\kappa$. First if $\alpha\in S_\mu$, by (9), then $\alpha\in\rho$, a contradiction. Next, suppose $\alpha\notin S_\mu$. Since $i\in\eta_\mu$, by (8), $a_i\subseteq S_\mu$ and so $\alpha\in S_\mu$, another contradiction. Therefore $\langle n,\alpha\rangle\in\sigma_\mu$. We have proved (10).

Since for $\mu<\nu<\omega_1$, r_μ and r_ν are incompatible, and by (10) $r_\mu\leq r_{\mu,\nu}$, it follows that $r_\nu\not\leq r_{\mu,\nu}$. This means

(11) for every $\nu>\mu$ there are $k_\nu\in\delta$, $i_\nu\in\eta_\nu$ and $\alpha\in a_{i_\nu}$ such that $\langle k_\nu,\alpha\rangle\in\sigma_\mu$ and $\langle k_\nu,\alpha\rangle\notin\sigma_\nu$.

By (6) and (11), we have $\alpha\in\rho_\mu\backslash\rho$, implying that (4) does not occur, and thus we are in Case (5).

Next let $\psi(\mu,\nu,k,i,\alpha)$ be the formula

$$\psi(\mu,\nu,k,i,\alpha)\equiv(k\in\delta)\wedge(i\in\eta)\wedge(\alpha\in a_i)\wedge(\langle k,\alpha\rangle\in\sigma_\mu)\wedge(\langle k,\alpha\rangle\notin\sigma_\nu)$$

and $I=\{\mu<\omega_1 : (\exists\nu)(\exists k)(\exists i)(\exists\alpha)\ ((\nu{>}\mu)\wedge\psi(\mu,\nu,k,i,\alpha))\}$. We claim that $|I|\leq\aleph_0$. Indeed, let $a:=\bigcup_{i\in\eta}a_i$, so $|a|=\aleph_0$. If $\nu>\mu$ and $\psi(\mu,\nu,k,i,\alpha)$ holds, then $\alpha\in a$ and $\alpha\in\rho_\mu\setminus\rho$. Since the $\rho_\mu\setminus\rho$'s are pairwise disjoint, $|I|\leq|a|=\aleph_0$.

Hence, for $\mu > \sup(I)$: if $\nu > \mu$, there is no i such that $\psi(\mu,\nu,k,i,\alpha)$ hold. This means that in (11), $i_\nu \in \eta_\nu \setminus \eta$. In particular $\eta_\nu \setminus \eta \neq \varnothing$. Hence we are in Case (2), that is,

(12) for every $\nu > \mu$ there are $k \in \delta$, $i_\nu \in \eta_\nu \backslash \eta$ and $\alpha \in a_{i_\nu}$ such that $\langle k,\alpha\rangle \in \sigma_\mu$ and $\langle k,\alpha\rangle \notin \sigma_\nu$. In particular $\alpha \in \rho_\mu \backslash \rho$.

We have proved that $\langle r_\mu : \mu < \omega_1\rangle$ satisfies (2), (3), (5) and (6)–(12).

By (5), let $p \in \omega$ be such that for every $\mu < \omega_1$, $|\rho_\mu \backslash \rho| := p \geq 1$. For $\mu \in \aleph_1$, let $\{\alpha^0_\mu, \alpha^1_\mu, \dots, \alpha^{p-1}_\mu\}$ be an enumeration of $\rho_\mu \backslash \rho \subseteq \kappa$. Also, by (2), for every $\mu, \nu < \omega_1$, $|\eta_\mu \backslash \eta| = |\eta_\nu \backslash \eta| := q \geq 1$. Let $\{i^0_\nu, i^1_\nu, \dots, i^{q-1}_\nu\}$ an enumeration of $\eta_\nu \backslash \eta$.

Let $m < n < \omega$. We define a color $c(m,n)$. An unordered pair may have more than one color. (This kind of coloring is used in the proof of Fact 6 in [2, p. 13].) Since $n < \omega + m$, r_n and $r_{\omega+m}$ are incompatible. By (10), $r_{\omega+m} \not\leq r_{n,\omega+m}$. We set $c(m,n) := \langle j,\ell\rangle$ if $i^j_{\omega+m} \in \eta_{\omega+m}$, $\alpha^\ell_n \in a_{i^j_{\omega+m}}$ (in particular $\alpha^\ell_n \in \rho_n$), there is $k \in \delta$ such that $\langle k, \alpha^\ell_n\rangle \in \sigma_n$, and $\langle k, \alpha^\ell_n\rangle \notin \sigma_{\omega+m}$. By (5) and (12), $c(m,n)$ exists. Also, since $j < q$ and $\ell < p$, the set of possible $\langle j,\ell\rangle$ is finite. By Ramsey's Theorem, let A be an infinite subset of ω and let $\langle j,\ell\rangle$ be such that for every distinct $m,n \in A$, if $m < n$ then $c(m,n) = \langle j,\ell\rangle$.

Let $m_0 < m_1$ be the first two members of A. Let $n \in A$ be such that $n > m_1$. Then $c(m_0,n) = c(m_1,n) = \langle j,\ell\rangle$. Hence $\alpha^\ell_n \in a_{i^j_{\omega+m_0}} \cap a_{i^j_{\omega+m_1}}$. Since $\alpha^\ell_n \in \rho_n \backslash \rho$ for $n \in \omega$, $\alpha^\ell_n \neq \alpha^\ell_{n'}$ for distinct members n, n' of A greater than m_1. Hence

(13) $a_{i^j_{\omega+m_0}} \cap a_{i^j_{\omega+m_1}}$ is infinite.

From (12), $i^j_{\omega+m_0}, i^j_{\omega+m_1} \notin \eta$, and thus $i^j_{\omega+m_0} \neq i^j_{\omega+m_1}$. By almost disjointness, $a_{i^j_{\omega+m_0}} \cap a_{i^j_{\omega+m_1}}$ is finite, which contradicts (13). We have proved Claim 3. $\square$

Now, by Martin's Axiom, let $G \subseteq \mathbb{P}$ be a filter which intersects each D_α and each E_i. For $n \in \omega$, let

$$p_n := \{\alpha : \text{for some } \langle \sigma,\eta\rangle \in G, \langle n,\alpha\rangle \in \sigma\}.$$

Obviously $\{p_n : n \in \omega\}$ is a partition of κ. Next, let $n \in \omega$ and $i \in \lambda$. Let $\langle \sigma,\eta\rangle \in G$ be such that $i \in \eta$ and $\langle n,\beta\rangle \in \sigma$ for some $\beta \in \kappa$. From the definition of $\leq$ on $\mathbb{P}$, it follows that $\{n\} \times a_i \subseteq \sigma$. Since σ is finite, $p_n \cap a_i$ is finite. So $\{p_n : n \in \omega\}$ is as required in the theorem. $\square$

Remark: (1) Assume Martin's Axiom and $2^{\aleph_0} > \aleph_1$. Let κ be a cardinal such that $\aleph_0 < \kappa < 2^{\aleph_0}$. Let $\mathcal{A}$ be a maximal almost disjoint family of countable subsets of κ. Note that $|\mathcal{A}| = 2^\kappa = 2^{\aleph_0}$ (see Theorem 2.18 and Corollary 2.16 of [4]). Next $\mathcal{A}$ does not satisfy the conclusion of Theorem 1.1. This is because for any partition $\{p_n : n \in \omega\}$ of κ, there is $m \in \omega$ such that $|p_m| > \aleph_0$; by the maximality of $\mathcal{A}$, for every $n \in \omega$ such that $|p_n| \geq \aleph_0$, there is $a \in \mathcal{A}$ such that $a \cap p_n$ is infinite.

(2) If $\kappa = \aleph_0$, then $\{\{n\} : n \in \aleph_0\}$ is a countable partition of $\aleph_0$, and trivially Theorem 1.1 holds for any family $\mathcal{A}$ of subsets of $\aleph_0$.

(3) In the conclusion of Theorem 1.1, we can replace the existence of a countable partition $\{p_n : n \in \omega\}$ of κ by the existence of a partition $\{p_m : m \in \rho\}$ where ρ is any infinite cardinal less or equal to κ.

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