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BOUNDS ON THE ACHROMATIC NUMBER OF PARTIAL TRIPLE SYSTEMS

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Abstract. A complete k-colouring of a hypergraph is an assignment of k colours to the points such that (1) there is no monochromatic hyperedge, and (2) identifying any two colours produces a monochromatic hyperedge. The achromatic number of a hypergraph is the maximum k such that it admits a complete k-colouring. We determine the maximum possible achromatic number among all maximal partial triple systems, give bounds on the maximum and minimum achromatic numbers of Steiner triple systems, and present a possible connection between optimal complete colourings and projective dimension.

1. Introduction

A *t-uniform hypergraph* is a pair $(V, \mathcal{A})$, where V is a nonempty set of *points* and $\mathcal{A}$ is a collection of t-subsets of V. Here, elements of $\mathcal{A}$ are called *blocks*. A *partial triple system* of *order* v, abbreviated PTS(v), is a 3-uniform hypergraph $(V, \mathcal{B})$ with $|V| = v$ and for which every pair of distinct elements of V is contained in at most one block. If every pair of distinct points in V occurs in exactly one block, then $(V, \mathcal{B})$ is a *Steiner triple system*, or STS(v). It is well-known that an STS(v) exists if and only if v is a positive integer with $v \equiv 1, 3 \pmod 6$. In the case $v = 1$, we take $\mathcal{B} = \emptyset$ to satisfy the conditions.

The *leave* of a PTS(v) $(V, \mathcal{B})$ is the graph (V, E) with xy an edge if and only if x and y are together in no block of $\mathcal{B}$. A PTS(v) is *maximal* if its leave is triangle-free. A consequence of Mantel's famous theorem on triangle-free graphs is that a maximal PTS(v) has at least $v(v-2)/12$ blocks. A PTS(u) $(U, \mathcal{A})$ is a *subsystem* of (or *embeds* in) a PTS(v) $(V, \mathcal{B})$ if $U \subseteq V$ and $\mathcal{A} \subseteq \mathcal{B}$. For later use, we state a recent result on embedding PTS into STS.

Lemma 1.1 ([1]). *Suppose $v > 2u$ and $v \equiv 1, 3 \pmod 6$. Then every PTS(u) is a subsystem of some STS(v).*

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A *proper k-colouring* of a t-uniform hypergraph $(V, \mathcal{A})$ is a mapping $c : V \to K$, where $|K| = k$, such that $|c(A)| > 1$ for each $A \in \mathcal{A}$. The elements of K are called *colours*, and we may assume that $K = \{1, \ldots, k\}$. The ith *colour class* is $c^{-1}(i)$, the set of points assigned colour i.

A proper k-colouring c is *complete* if for every pair i, j of distinct colours there is a block $A \in \mathcal{A}$ with $c(A) = \{i, j\}$. We say that the pair $\{i, j\}$ is *covered by A*. The *achromatic number* of $(V, \mathcal{A})$, denoted $\psi(\mathcal{A})$, is the maximum k such that $(V, \mathcal{A})$ admits a complete k-colouring.

For each positive integer n, define $\psi_{\min}(v)$ and $\psi_{\max}(v)$ as the minimum and maximum achromatic numbers, respectively, of a maximal PTS(v). For $v \equiv 1, 3 \pmod 6$, define $\psi^*_{\min}(v)$ and $\psi^*_{\max}(v)$ similarly for STS(v). When meaningful, it is clear that

$$\psi_{\min}(v) \le \psi^*_{\min}(v) \le \psi^*_{\max}(v) \le \psi_{\max}(v).$$

In the next section, we give an explicit upper bound on $\psi_{\max}(v)$. Then, in section 3, we show that this upper bound is met with equality by constructing a PTS(v) with this achromatic number. Bounds for $\psi^*_{\max}(v)$ follow as a consequence. In section 4, we modify an argument in [3] to establish lower bounds on $\psi_{\min}(v)$ and $\psi^*_{\min}(v)$. We present upper bounds and open problems concerning these quantities in section 5. Finally, various optimal complete colourings of small STS are compiled in section 6.

2. Upper bounds on $\psi_{\max}$

In [3], it was shown that $\psi_{\max}(v)$ is $O(v^{2/3})$. With straightforward counting, we are able to obtain an exact upper bound.

Lemma 2.1. *If there exists a complete k-colouring of a PTS(v) with colour class sizes $y_1 \le y_2 \le \cdots \le y_k$, then $\sum_{i=1}^{j} \binom{y_i}{2} \ge \binom{j}{2}$ for all $j = 1, \ldots, k$.*

Proof. Given a complete k-colouring $c : V \to K$ of $(V, \mathcal{B})$, define a digraph D with vertex set K and with (i, j) an arc if and only if there exists a block $B \in \mathcal{B}$ with $|c^{-1}(i) \cap B| = 2$ and $|c^{-1}(j) \cap B| = 1$ (that is, if B has two points coloured i and one point coloured j). Let $s_1 \le s_2 \le \cdots \le s_k$ be the sequence of outdegrees of D. We have $s_i \le \binom{y_i}{2}$. By completeness, D (and each of its induced subdigraphs) is semi-complete. So $\sum_{i=1}^{j} \binom{y_i}{2} \ge \sum_{i=1}^{j} s_i \ge \binom{j}{2}$ for $1 \le j \le k$. □

Corollary 2.2. *In a complete k-colouring of a PTS(v), there are at most $t^2 - t + 1$ colour classes of size $\le t$.*

Proof. Suppose there are k colour classes of size $\le t$. By Lemma 2.1, $k\binom{t}{2} \ge \binom{k}{2}$, from which the result follows. □

Remark 2.3. *These results can be generalized for triple systems of higher* index λ *in which every pair of distinct points belongs to at most λ blocks.*

For $n \geq 1$, define $a_n = \lceil \sqrt{2n-1} \rceil$. The sequence $\{a_n\}$ begins

$$1,2,3,3,3,4,4,4,5,5,5,5,5,6,6,6,6,6,\ldots,\overbrace{2i-1,\ldots,2i-1}^{2i-1},\overbrace{2i,\ldots,2i}^{2i-1},\ldots.$$

The following facts are easily verified by induction. We omit the proofs.

(1) $\sum_{i=1}^{j} \binom{a_i}{2} \geq \binom{j}{2}$ for all j, with equality if and only if $j = 2n^2$ or $j = 2n^2 + 2n + 1$ for some n;

(2) if, for all $j = 1, \ldots, k$, we have $b_j \leq a_j$ and $\sum_{i=1}^{j} \binom{b_i}{2} \geq \binom{j}{2}$, then $b_j = a_j$ for all $j = 1, \ldots, k$.

Taken with Lemma 2.1, (1) and (2) lead to an upper bound on $\psi_{\max}$.

Theorem 2.4. *Let* $a_n = \lceil \sqrt{2n-1} \rceil$. *Then*

$$\psi_{\max}(v) \leq \max\{k : a_1 + a_2 + \cdots + a_k \leq v\}.$$

Proof. Let k denote the right side above, and assume there is a complete $(k+1)$-colouring of some PTS(v), say with colour class sizes $y_1 \leq y_2 \leq \cdots \leq y_{k+1}$. By Lemma 2.1, we have $\sum_{i=1}^{j} \binom{y_i}{2} \geq \binom{j}{2}$ for $j = 1, \ldots, k+1$. We repeatedly transform the sequence of y_i as follows. Since $\sum_{i=1}^{k+1} y_i < \sum_{i=1}^{k+1} a_i$, we may choose the smallest integer $m \in \{1, \ldots, k+1\}$ such that $y_m < a_m$. By (2) above, we cannot have $y_i = a_i$ for all $i < m$, so take the largest integer $n \in \{1, \ldots, m-1\}$ with $y_n > a_n$. Now define integers $w_1, \ldots, w_{k+1}$ with $w_n = y_n - 1$, $w_m = y_m + 1$, and $w_i = y_i$ for all $i \neq n, m$. Now property (1) above implies

$$\sum_{i=1}^{j} \binom{w_i}{2} \geq \sum_{i=1}^{j} \binom{a_i}{2} \geq \binom{j}{2}$$

for $j < m$ and since $y_m \geq y_n > w_n$, we have

$$\sum_{i=1}^{j} \binom{w_i}{2} = \sum_{i=1}^{j} \binom{y_i}{2} + (y_m - w_n) > \binom{j}{2}$$

for $j \geq m$. Now relabel each w_i as y_i' and choose indices m' and n' for y_i' as m and n were chosen for y_i. It is noteworthy that $y_{m'}' \geq y_{m'} \geq y_{n'} \geq y_{n'}'$, so the estimate on $\sum_{i=1}^{j} \binom{w_i}{2}$ for $j \geq m$ remains valid in subsequent steps. This process must terminate, contradicting (2). Therefore, the supposed colouring does not exist. □

It is particularly interesting to consider the case of equality in (1). For $k = 2n^2$, we calculate

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{n} (2i-1)(4i-1) = \frac{8}{3}n^3 + n^2 - \frac{2}{3}n,$$

and similarly, for $k = n^2 + (n+1)^2$,

$$\sum_{i=1}^{k} a_i = \frac{8}{3}n^3 + 5n^2 + \frac{10}{3}n + 1.$$

For future reference, we label these cubic polynomials in n as $p_1(n)$ and $p_2(n)$, respectively. The following consequence of (1), (2), and Theorem 2.4 is now immediate.

Corollary 2.5. *Let $k = 2n^2$, (or $k = n^2 + (n+1)^2$), where n is a positive integer. If there is a complete k-colouring of some PTS(v), then $v \geq p_1(n)$ (respectively $v \geq p_2(n)$), with equality if and only if the colour class sizes are exactly $a_1, a_2, \ldots, a_k$.*

By Corollary 2.5 and condition (1) on the sequence $\{a_n\}$, when $v = p_1(n)$ or $v = p_2(n)$, any $\psi_{\max}(v)$-colouring of a PTS(v) uses every pair of points within a colour class to cover some pair of colours, and that each pair of colours is covered exactly once. We consider an application of this structure in section 5.

3. Lower bounds on $\psi_{\max}$ and $\psi^*_{\max}$

We begin with a construction of a family of PTS having largest possible achromatic number.

Theorem 3.1. *Let $a_n = \left\lceil \sqrt{2n-1} \right\rceil$. Then*

$$\psi_{\max}(v) \geq \max\{k : a_1 + a_2 + \cdots + a_k \leq v\}.$$

Proof. For $h > 1$, define $[h] = \max\{l : a_l < a_h\}$. We construct a PTS on points $X_1 \cup \cdots \cup X_k$, where the X_i are pairwise disjoint colour classes with $|X_i| = a_i$, and every pair of colours is covered by some block. It is sufficient to perform this construction for an infinite sequence of k. Hence, we will assume $[k+1] = k$, by replacing k by the least integer K with $[K] > [k]$, and deleting if necessary the points in $X_{k+1} \cup \cdots \cup X_K$. For $j > 1$, order the pairs in X_j arbitrarily. Form blocks by joining the ith pair in X_j to some point in X_i, for each $i = 1, \ldots, [j]$. Note that

$$\binom{a_j}{2} - [j] = \lceil a_j/2 \rceil - 1,$$

so every pair of colours $i < j$ with $a_i < a_j$ is now covered by some block. For a given j with $1 < j \leq k$, there are $b_j = 2\lceil a_j/2 \rceil - 1$ values of i with $[i] = [j]$, or $a_i = a_j$. It remains to define blocks covering the pairs i, j, with $[i] < i < j \leq [i] + b_j$. Consider the complete graph K_{b_j} on vertices $1, \ldots, b_j$. Orient its edges such that every vertex has indegree and outdegree equal to $\lceil a_j/2 \rceil - 1$. (For instance, this can be done using a decomposition into Hamilton cycles.) For each edge directed from r to s, form a block by joining a unique choice of one of the remaining $\binom{a_j}{2} - [j]$ pairs of $X_{[i]+r}$ to some point in $X_{[i]+s}$. □

Corollary 3.2.

$$\psi_{\max}(v) = \max\{k : a_1 + a_2 + \cdots + a_k \leq v\}.$$

Proof. This follows directly from Theorems 2.4 and 3.1. □

The asymptotic behavior of $\psi_{\max}$ is easily calculated from the definition of a_n.

Corollary 3.3.

$$\lim_{v\to\infty} \frac{\psi_{\max}(v)}{v^{2/3}} = \frac{1}{2} \cdot 3^{2/3}.$$

We now turn to the question of lower bounds on $\psi^*_{\max}(v)$, the maximum achromatic number of a Steiner triple system of order v.

Suppose a PTS $(U, \mathcal{A})$ is a subsystem of another PTS $(V, \mathcal{B})$. Let us call a colouring of $(U, \mathcal{A})$ *safe with respect to* $\mathcal{B}$ if it induces no monochromatic block in $\mathcal{B}$.

Lemma 3.4. *Suppose $(U, \mathcal{A})$ is a PTS(u) with a complete k-colouring that is safe with respect to some $\mathcal{B} \supseteq \mathcal{A}$. Then a PTS$(v)$ $(V, \mathcal{B})$ has a complete l-colouring for some $l \geq k$.*

Proof. Consider a PTS(v) $(V, \mathcal{B})$ with $U \subseteq V$. Initially colour the points of U with a complete k-colouring, safe with respect to $\mathcal{B}$, using colours $1, \dots, k$, and the points of $V \setminus U$ each with a distinct new colour, using colours $k+1, \dots, k+v-u$. Since the colouring of U is safe, it follows that this colouring of V is proper. Now repeatedly perform the following operation: merge any two colour classes $i < j$ which are not covered by any block, and rename this colour class i. It is clear that classes $1, \dots, k$ remain nonempty, and when no merging is possible, a complete colouring results. □

It is evident that the construction in Theorem 3.1 can be done in such a way that the resulting colouring is safe with respect to any $\mathcal{B} \supseteq \mathcal{A}$ (for instance, if the u "unused" pairs of points in a colour class are chosen to form a star $K_{1,u}$). Together with Lemmas 1.1 and 3.4, we obtain the following result.

Theorem 3.5.

$$\psi^*_{\max}(v) \geq \max\{k : a_1 + a_2 + \cdots + a_k < v/2\}.$$

Corollary 3.6. *If $\lim_{v\to\infty} \psi^*_{\max}(v)/v^{2/3}$ exists and equals L, then*

$$\frac{1}{2} \cdot (3/2)^{2/3} \leq L \leq \frac{1}{2} \cdot 3^{2/3}.$$

In practice, it seems easy to embed *some* PTS(v) from Theorem 3.1 into *some* STS(v), provided $v \equiv 1, 3 \pmod 6$. Using a standard hill-climbing algorithm for completing STS, we were able to do so for all "small" values of v. Some specific constructions are given in the appendix.

Theorem 3.7. *If $v \equiv 1, 3 \pmod 6$ and $1 \leq v \leq 49$, then $\psi^*_{\max}(v) = \psi_{\max}(v)$.*

An interesting question is whether such an embedding is always possible. If so, the following would be proved.

Conjecture 3.8. *For all $v \equiv 1, 3 \pmod 6$, $\psi^*_{\max}(v) = \psi_{\max}(v)$.*

4. Lower bounds on $\psi_{\min}$ and $\psi^*_{\min}$

There is a unique STS(v) up to isomorphism for $v \leq 9$. So $\psi^*_{\min}(v) = \psi^*_{\max}(v)$ for $v \leq 9$. For $v = 13, 15$, there are, respectively, 2 and 80 different STS up to isomorphism. After a very fast computer search, we report that every STS(13) and STS(15) admits a complete 5-colouring.

Theorem 4.1. *$\psi^*_{\min}(v) = 5$ for $v = 13, 15$.*

The following is adapted from [3].

Theorem 4.2. *Any PTS with minimum degree $t \geq 3$ admits a complete k-colouring, where*

$$k = \left\lfloor \frac{11 + \sqrt{12t - 11}}{6} \right\rfloor.$$

Proof. Let $(V, \mathcal{B})$ be such a PTS(v). Since $t \geq 3$, we have $2 \leq k < t$. Take $x_1 \in V$ and blocks $B_1, B_2, \dots, B_{k-1}$, each containing x_1. Define $X_1 = \{x_1\}$, $X_{1,i} = B_i \setminus \{x_1\}$ for each $i = 1, 2, \dots, k-1$. Suppose for some $r \geq 1$ we have constructed pairwise disjoint sets

$$X_1, X_2, \dots, X_r, X_{r,r}, X_{r,r+1}, \dots, X_{r,k-1} \subset V,$$

where $|X_1| = 1, |X_i| = 2i - 2$ for $i = 2, 3, \dots, r$, and $|X_{r,j}| = 2r$ for $j = r, r+1, \dots, k-1$. Further suppose that

(1) for any $i, j \leq r$ with $i \neq j$, $X_i \cup X_j$ contains a block of $\mathcal{B}$;
(2) for any $i \leq r$ and $j \geq r$, $X_i \cup X_{r,j}$ contains a block of $\mathcal{B}$; and
(3) there does not exist i such that X_i or $X_{r,i}$ contains a block of $\mathcal{B}$.

Evidently, these conditions hold for $r = 1$. If $r = k-1$, define $X_k = X_{k-1,k-1}$ and the construction is done. The sets $X_1, \dots, X_k$ are colour classes of a complete k-colouring of some subsystem $(U, \mathcal{A})$ which is safe with respect to $\mathcal{B}$. Lemma 3.4 extends this colouring to $(V, \mathcal{B})$. It remains to show that for $r < k - 1$ we can continue the construction.

Take a point $x_{r+1} \in X_{r,r}$ and define

$$W_r = \left(\bigcup_{i=1}^{r} X_i \right) \cup \left(\bigcup_{j=r}^{k-1} X_{r,j} \right),$$

the set of all points used in the construction so far. There are at most $|W_r| - 1 = r(r-1) + 2r(k-r)$ blocks of the form $\{x_{r+1}, y, w\}$ with $w \in W_r$.

Now for $i = r+1, \dots, k-1$, let $Z_{r,i}$ be the set of points $z \notin W_r$ such that for some $B \in \mathcal{B}$, $z \in B$ and $B \setminus \{z\} \subseteq X_{r,i}$. For such values of i, we are forbidden by (3) above to extend $X_{r,i}$ to $X_{r+1,i}$ by adding a point in $Z_{r,i}$. Note that $|Z_{r,i}| \leq \binom{2r}{2} - r = 2r(r-1)$ for each i, since there are this many "unused" pairs in $X_{r,i}$.

It follows that there are at least $t - 3r(r-1) - 2r(k-r)$ choices for a block B_i containing x_{r+1} with $B_i \setminus x_{r+1}$ disjoint from W_r and extending $X_{r,i}$

without violating (3). Since we must pick such blocks for $i = r+1, \ldots, k-1$, the construction can continue if and only if

$$t - 3r(r-1) - 2r(k-r) \geq k - r - 1,$$

or equivalently

$$r^2 + 2kr - 4r + k - 1 \leq t.$$

The quadratic in r on the left is minimized for $r = 2 - k$. For our purposes, $1 \leq r < k - 1$, so the left side is maximized for $r = k - 2$. The condition becomes $3k^2 - 11k + 11 \leq t$, or

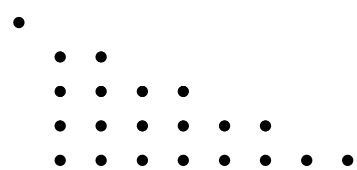
$$k \leq \frac{11 + \sqrt{12t - 11}}{6}.$$

Having chosen distinct blocks $B_i \in \mathcal{B}$ which contain x_{r+1} and no other point of $W_r \cup Z_{r,i}$, we put

$$X_{r+1} = X_{r,r} \text{ and } X_{r+1,i} = X_{r,i} \cup (B_i \setminus \{x_{r+1}\})$$

for $i = r+1, \ldots, k-1$. Observe that properties (1), (2), and (3) above hold for $X_1, \ldots, X_{r+1}, X_{r+1,r+1}, \ldots, X_{r+1,k-1}$. □

We remark that the argument proving Theorem 4.2 can be slightly improved with a careful choice of points and colour classes. We omit the details here. The construction of $X_1, \ldots, X_k$ is illustrated for $k = 5$ below. The ith row represents X_i, with the rightmost point representing a choice of x_i. Blocks joining this point to pairs across the next two columns are present in $\mathcal{B}$.



Corollary 4.3. *For $v \geq 7$,*

$$\psi^*_{\min}(v) \geq \frac{11 + \sqrt{6v - 17}}{6}.$$

Proof. Let $t = \frac{v-1}{2}$ in Theorem 4.2. □

Corollary 4.4. *Suppose $v > 24$. Then*

$$\psi_{\min}(v) \geq \frac{11 + \sqrt{v - 11}}{6}.$$

Proof. It is enough to show that any maximal PTS(v) $(V, \mathcal{B})$ with $v > 24$, has a sub-PTS with the minimum degree of any point at least $v/12$. Choose $x \in V$ with degree $< v/12$, and delete it together with all blocks through it. Repeat this process until all points have degree $\geq v/12$. We claim that this process terminates with at least $(v-5)/2$ points left. Suppose $\lfloor (v+5)/2 \rfloor$ points, each of degree $< v/12$, have been removed. Then at least $v(v-2)/12 - v(v+5)/24 = v(v-9)/24$ blocks remain. But there can be no more than $\frac{1}{3}\binom{(v-4)/2}{2} = (v-4)(v-6)/24$ blocks left. This contradicts $v > 24$. □

5. Upper bounds on $\psi_{\min}$

It is natural to ask whether any maximal PTS(v) has achromatic number less than $\psi_{\max}(v)$. The following gives an infinite family of such PTS.

Theorem 5.1. *Suppose $v = p_1(n) = \frac{8}{3}n^3 + n^2 - \frac{2}{3}n$ or $v = p_2(n) = \frac{8}{3}n^3 + 5n^2 + \frac{10}{3}n + 1$, where $n \geq 2$ is an integer. Then $\psi_{\min}(v) < \psi_{\max}(v)$.*

Proof. Assume first that v is even. Since $v \geq 24$, we can write $v = u + u'$, where $u, u' \equiv 1$ or $3 \pmod 6$ and $u, u' \geq 9$. Define a PTS(v) $(V, \mathcal{B})$, where $\mathcal{B}$ is the union of blocks of an STS(u) and an STS(u') on points U, U', where $U \cap U' = \emptyset$. It is clear that $(V, \mathcal{B})$ is a maximal PTS(v). If $(V, \mathcal{B})$ were to admit a complete $\psi_{\max}(v)$-colouring, then by the discussion following Corollary 2.5, every colour class is either completely in U or completely in U'. Since $v \geq 24$, there is certainly a pair of colours uncovered by $\mathcal{B}$. This is a contradiction, and therefore $\psi_{\min}(v) < \psi_{\max}(v)$. The case when $v(\geq 49)$ is odd is similar, except we write $v = u + u' - 1$ and $V = U \cup U'$, where $|U \cap U'| = 1$. □

The question of whether $\psi^*_{\min}(v) < \psi^*_{\max}(v)$ for any v seems more difficult. One approach would be to attempt a construction of STS(v) avoiding certain configurations of blocks required in an optimal colouring. Although we do not have an example of an STS(v) with a provably "bad" achromatic number, there is an infinite family of STS(v) for which we can deduce information about optimal colourings.

Given a PTS(v), say $(V, \mathcal{B})$, we say $I \subset V$ is an *independent set* if there is no $B \in \mathcal{B}$ with $B \subseteq I$. A complete k-colouring of a PTS(v) is equivalent to a partition $I_1, \ldots, I_k$ of V into independent sets such that $I_i \cup I_j$ is not independent for $i \neq j$. From this observation and the pigeonhole principle, we obtain a structural result on colour class sizes.

Theorem 5.2. *Let $\mathcal{I} = \{I_1, \ldots, I_N\}$ be a family of independent sets of a PTS(v). Suppose for every positive integer i that each independent i-subset of V is contained in at least m_i elements of $\mathcal{I}$. In any complete k-colouring with n_i colour classes of size i, we have*

$$\sum_{i \geq 1} m_i n_i \leq N.$$

Let $(V, \mathcal{B})$ be a fixed STS(v). Any STS($(v-1)/2$), $(U, \mathcal{A})$, which is a subsystem is called a *projective hyperplane*. Note that for such U, $V \setminus U$ is necessarily a maximal (in fact, maximum) independent set.

For $X \subset V$, let $\mathbf{e}_X : V \to \{0, 1\}$ denote the *characteristic vector* of X, where $\mathbf{e}_X(x) = 1$ if and only if $x \in X$. The following is an observation of Teirlinck.

Theorem 5.3 ([4]). *Let $\mathcal{I}$ be the set of all $I \subseteq V$, in which $V \setminus I$ induces a projective hyperplane of $(V, \mathcal{B})$. Then $\mathcal{W} = \{\mathbf{e}_I : I \in \mathcal{I}\} \cup \{\mathbf{0}\}$ is a vector space over $\mathbb{F}_2$.*

Throughout, let d denote the dimension of $\mathcal{W}$. A well-known example of an STS whose independent sets induce a vector space of dimension d is now given. Let $V = \mathbb{F}_2^{d+1} \setminus \{\mathbf{0}\}$, the set of all nonzero binary $(d+1)$-tuples. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{V}$, define $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \in \mathcal{B}$ if and only if $\mathbf{x} + \mathbf{y} = \mathbf{z}$. It is easy to see that $(V, \mathcal{B})$ is an STS(v), where $v = 2^{d+1} - 1$, called the *projective* STS of *dimension d*.

The following facts are easily verified with linear algebra.

Lemma 5.4. *In the projective STS of dimension d, any independent t-subset T of points of V, $0 \le t \le d+1$, is contained in exactly $2^{d+1-t} - 1$ projective hyperplanes, and is disjoint from exactly 2^{d+1-t} projective hyperplanes for $t \ge 1$.*

By Theorem 5.2, we can now make a statement about colourings of projective STS.

Corollary 5.5. *Suppose there exists a complete colouring of the projective STS of dimension d with n_i colour classes of size i, $i = 1, \ldots, d+1$. Then*

$$\sum_{i=1}^{d+1} 2^{d+1-i} n_i \le 2^{d+1} - 1.$$

It is perhaps unfortunate that the Diophantine equations $p_1(n) = 2^d - 1$ and $p_2(n) = 2^d - 1$ each have no solutions, as Corollaries 2.5 and 5.5 would lead to the conclusion that certain projective STS(v) have "bad" achromatic numbers. It appears that projective dimension is worth further attention in future work on $\psi^*_{\min}(v)$.

6. Appendix: Complete colourings of small STS

6.1. Direct colourings.

- $v = 7$, $\psi_{\max} = 3$
  System: $\{0,1,2\}$, $\{0,3,4\}$, $\{0,5,6\}$, $\{1,3,5\}$, $\{1,4,6\}$, $\{2,3,6\}$, $\{2,4,5\}$
  Colouring: $0 \mapsto 1$, $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 3$, $4 \mapsto 3$, $5 \mapsto 1$, $6 \mapsto 2$
- $v = 9$, $\psi_{\max} = 4$
  System: $\{0,1,2\}$, $\{0,3,6\}$, $\{0,4,8\}$, $\{0,5,7\}$, $\{1,3,8\}$, $\{1,4,7\}$, $\{1,5,6\}$, $\{2,3,7\}$, $\{2,4,6\}$, $\{2,5,8\}$, $\{3,4,5\}$, $\{6,7,8\}$,
  Colouring: $0 \mapsto 1$, $1 \mapsto 2$, $2 \mapsto 2$, $3 \mapsto 1$, $4 \mapsto 3$, $5 \mapsto 4$, $6 \mapsto 3$, $7 \mapsto 4$, $8 \mapsto 4$
- $v = 13$, $\psi_{\max} = 5$
  System: $\{0,1,4\}$, $\{1,2,5\}$, $\{2,3,6\}$, $\{3,4,7\}$, $\{4,5,8\}$, $\{5,6,9\}$, $\{6,7,10\}$, $\{7,8,11\}$, $\{8,9,12\}$, $\{9,10,0\}$, $\{10,11,1\}$, $\{11,12,2\}$, $\{12,0,3\}$, $\{0,2,7\}$, $\{1,3,8\}$, $\{2,4,9\}$, $\{3,5,10\}$, $\{4,6,11\}$, $\{5,7,12\}$, $\{6,8,0\}$, $\{7,9,1\}$, $\{8,10,2\}$, $\{9,11,3\}$, $\{10,12,4\}$, $\{11,0,5\}$, $\{12,1,6\}$
  Colouring: $0 \mapsto 2$, $1 \mapsto 4$, $2 \mapsto 4$, $3 \mapsto 4$, $4 \mapsto 5$, $5 \mapsto 3$, $6 \mapsto 3$, $7 \mapsto 3$, $8 \mapsto 5$, $9 \mapsto 2$, $10 \mapsto 1$, $11 \mapsto 4$, $12 \mapsto 5$

- $v = 15$, $\psi_{\max} = 5$
  System: $\{0,1,2\}$, $\{0,3,7\}$, $\{0,4,12\}$, $\{0,5,8\}$, $\{0,6,11\}$, $\{0,9,13\}$, $\{0,10,14\}$, $\{3,4,5\}$, $\{1,4,10\}$, $\{1,7,13\}$, $\{1,9,14\}$, $\{1,3,9\}$, $\{1,8,12\}$, $\{1,5,6\}$, $\{6,7,8\}$, $\{2,8,14\}$, $\{2,6,10\}$, $\{2,7,9\}$, $\{2,5,12\}$, $\{2,4,11\}$, $\{2,3,13\}$, $\{9,10,11\}$, $\{5,9,13\}$, $\{3,8,11\}$, $\{3,10,12\}$, $\{4,7,14\}$, $\{3,14,6\}$, $\{4,8,9\}$, $\{12,13,14\}$, $\{6,9,12\}$, $\{5,9,14\}$, $\{4,6,13\}$, $\{8,10,13\}$, $\{5,7,10\}$, $\{7,11,12\}$
  Colouring: $0 \mapsto 1$, $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$, $4 \mapsto 2$, $5 \mapsto 4$, $6 \mapsto 4$, $7 \mapsto 3$, $8 \mapsto 3$, $9 \mapsto 4$, $10 \mapsto 5$, $11 \mapsto 4$, $12 \mapsto 3$, $13 \mapsto 1$, $14 \mapsto 1$

6.2. **Computer constructions.** In each example below, a PTS is given from the proof of Theorem 4.2. A simple hill-climbing algorithm embeds each PTS in an STS(v). Each computation took a few seconds on a personal computer.

- $v = 19$, $\psi_{\max} = 6$
  Forcing PTS:
  $\{0,1,2\}$, $\{0,3,4\}$, $\{1,3,5\}$, $\{4,5,6\}$, $\{0,6,7\}$, $\{1,6,8\}$, $\{7,8,9\}$, $\{0,9,10\}$, $\{1,9,11\}$, $\{3,10,11\}$, $\{0,12,13\}$, $\{1,12,14\}$, $\{5,13,14\}$, $\{7,12,15\}$, $\{10,13,15\}$
  Embedding:
  $\{2,3,13\}$, $\{4,9,15\}$, $\{3,7,16\}$, $\{5,7,18\}$, $\{1,4,10\}$, $\{2,12,18\}$, $\{15,17,18\}$, $\{5,8,16\}$, $\{3,8,18\}$, $\{9,14,18\}$, $\{11,13,16\}$, $\{8,11,15\}$, $\{1,13,18\}$, $\{4,7,13\}$, $\{4,11,18\}$, $\{2,4,8\}$, $\{3,9,17\}$, $\{0,16,18\}$, $\{1,15,16\}$, $\{6,10,18\}$, $\{3,6,12\}$, $\{8,10,12\}$, $\{2,6,15\}$, $\{6,11,14\}$, $\{2,5,9\}$, $\{2,7,11\}$, $\{9,12,16\}$, $\{5,10,17\}$, $\{8,13,17\}$, $\{1,7,17\}$, $\{0,5,15\}$, $\{2,10,16\}$, $\{6,16,17\}$, $\{6,9,13\}$, $\{0,11,17\}$, $\{5,11,12\}$, $\{2,14,17\}$, $\{7,10,14\}$, $\{4,12,17\}$, $\{4,14,16\}$, $\{0,8,14\}$, $\{3,14,15\}$
- $v = 21$, $\psi_{\max} = 7$
  Forcing PTS:
  $\{0,1,2\}$, $\{0,3,4\}$, $\{1,3,5\}$, $\{4,5,6\}$, $\{0,6,7\}$, $\{1,6,8\}$, $\{7,8,9\}$, $\{0,9,10\}$, $\{1,9,11\}$, $\{3,10,11\}$, $\{0,12,13\}$, $\{1,12,14\}$, $\{5,13,14\}$, $\{7,12,15\}$, $\{10,13,15\}$, $\{14,15,16\}$, $\{0,16,17\}$, $\{2,16,18\}$, $\{4,17,18\}$, $\{7,16,19\}$, $\{11,17,19\}$
  Embedding:
  $\{5,7,20\}$, $\{0,8,18\}$, $\{9,13,18\}$, $\{9,12,16\}$, $\{5,15,18\}$, $\{4,19,20\}$, $\{2,6,15\}$, $\{1,10,18\}$, $\{3,6,16\}$, $\{9,14,17\}$, $\{14,18,19\}$, $\{8,15,17\}$, $\{2,10,19\}$, $\{6,10,17\}$, $\{5,8,11\}$, $\{11,13,16\}$, $\{3,15,20\}$, $\{2,3,12\}$, $\{0,11,15\}$, $\{3,7,18\}$, $\{2,5,9\}$, $\{4,7,13\}$, $\{0,14,20\}$, $\{6,13,19\}$, $\{5,10,16\}$, $\{3,8,14\}$, $\{4,11,12\}$, $\{5,12,17\}$, $\{6,9,20\}$, $\{6,12,18\}$, $\{1,15,19\}$, $\{4,9,15\}$, $\{1,7,17\}$, $\{2,17,20\}$, $\{8,16,20\}$, $\{4,8,10\}$, $\{11,18,20\}$, $\{1,4,16\}$, $\{1,13,20\}$, $\{2,7,11\}$, $\{10,12,20\}$, $\{3,13,17\}$, $\{7,10,14\}$, $\{0,5,19\}$, $\{8,12,19\}$
- $v = 25$, $\psi_{\max} = 8$
  Forcing PTS:
  $\{0,1,2\}$, $\{0,3,4\}$, $\{1,3,5\}$, $\{4,5,6\}$, $\{0,6,7\}$, $\{1,6,8\}$, $\{7,8,9\}$, $\{0,9,10\}$, $\{1,9,11\}$, $\{3,10,11\}$, $\{0,12,13\}$, $\{1,12,14\}$, $\{5,13,14\}$, $\{7,12,15\}$, $\{10,13,15\}$, $\{14,15,16\}$, $\{0,16,17\}$, $\{2,16,18\}$, $\{4,17,18\}$, $\{7,16,19\}$, $\{11,17,19\}$, $\{18,19,20\}$, $\{0,20,21\}$, $\{2,20,22\}$, $\{4,21,22\}$, $\{6,20,23\}$, $\{10,21,23\}$, $\{12,22,23\}$
  Embedding:
  $\{0,14,19\}$, $\{2,3,9\}$, $\{3,8,13\}$, $\{2,4,23\}$, $\{5,11,16\}$, $\{7,11,21\}$, $\{5,7,22\}$, $\{8,12,17\}$, $\{8,11,23\}$, $\{0,15,23\}$, $\{3,17,24\}$, $\{8,19,24\}$, $\{6,13,17\}$, $\{3,15,20\}$, $\{1,13,22\}$, $\{10,17,22\}$, $\{9,13,20\}$, $\{1,10,16\}$, $\{9,14,18\}$, $\{2,7,14\}$, $\{5,8,20\}$, $\{2,5,19\}$, $\{8,15,18\}$, $\{2,8,10\}$, $\{15,22,24\}$, $\{3,16,22\}$, $\{2,15,17\}$, $\{1,18,23\}$, $\{10,18,24\}$, $\{8,16,21\}$, $\{16,20,24\}$, $\{6,12,16\}$, $\{3,14,21\}$, $\{11,13,24\}$, $\{3,7,23\}$, $\{2,12,24\}$, $\{14,17,20\}$, $\{1,15,19\}$, $\{11,12,20\}$, $\{0,5,24\}$,

$\{14, 23, 24\}$, $\{5, 17, 23\}$, $\{6, 18, 22\}$, $\{13, 19, 23\}$, $\{2, 6, 11\}$, $\{9, 19, 22\}$,
$\{4, 10, 19\}$, $\{1, 4, 20\}$, $\{7, 10, 20\}$, $\{12, 19, 21\}$, $\{5, 9, 15\}$, $\{4, 9, 12\}$, $\{6, 15, 21\}$,
$\{4, 7, 24\}$, $\{1, 7, 17\}$, $\{4, 8, 14\}$, $\{1, 21, 24\}$, $\{5, 10, 12\}$, $\{3, 12, 18\}$, $\{0, 8, 22\}$,
$\{4, 11, 15\}$, $\{3, 6, 19\}$, $\{4, 13, 16\}$, $\{6, 9, 24\}$, $\{9, 16, 23\}$, $\{5, 18, 21\}$, $\{9, 17, 21\}$,
$\{0, 11, 18\}$, $\{2, 13, 21\}$, $\{7, 13, 18\}$, $\{6, 10, 14\}$, $\{11, 14, 22\}$

6.3. **A recursive construction.** In some cases, recursive constructions of Steiner triple systems are amenable to complete colourings. We give one illustration of this.

Lemma 6.1. *Suppose there exists an STS*(u), $(U, \mathcal{A})$, *admitting a complete* k*-colouring with colour class sizes* $w_1, w_2, \dots, w_k$. *Let* $\sigma \in \mathcal{S}_k$ *be some permutation. Suppose there exists a Latin square* L *of side* n *with row and column-disjoint* $w_i \times w_{\sigma(i)}$ *sub-rectangles* R_i, *each of which contains the entries* $\{e_1, \dots, e_k\}$. *Then there exists an STS*$(3u)$ *admitting a complete* $2k$*-colouring.*

Proof. We apply the standard tripling construction for Steiner triple systems. Let $V = U \times \{1, 2, 3\}$, and let $U \times \{1\}$, $U \times \{2\}$, $U \times \{3\}$ index the rows, columns, and entries of L, respectively. Define a set of blocks $\mathcal{B}$ on V by including $A \times \{i\} \in \mathcal{B}$ for every $A \in \mathcal{A}$. In addition, put $\{x, y, L(x, y)\} \in \mathcal{B}$ for every $x \in U \times \{1\}$ and $y \in U \times \{2\}$. So $(V, \mathcal{B})$ is an STS$(3u)$, and we now describe a colouring of it. Colour the points in $U \times \{1\}$ with a complete k-colouring of $\{A \times \{1\} : A \in \mathcal{A}\}$, and such that rows in subrectangle R_i get colour i. Likewise, colour $U \times \{2\}$ so that columns of R_i receive colour i. Every pair of colours $\{i, j\}$, $1 \le i < j \le k$, is now covered (at least twice). Colour the points in $U \times \{3\}$ with a complete k-colouring of $\{A \times \{3\} : A \in \mathcal{A}\}$, using colours $\{k+1, \dots, 2k\}$. This covers pairs of colours $\{k + i, k + j\}$, where $1 \le i < j \le k$. We may arrange this latter colouring so that entry e_j receives colour $k + j$ for $j = 1, \dots, k$. By hypothesis, every pair of colours $\{i, k + j\}$, with $i, j \in \{1, \dots, k\}$, is covered by $\mathcal{B}$. □

Example 6.2. *Using Lemma 6.1, we have* $\psi_{\max}(27) = 8$. *Use the complete 4-colouring of the STS(9) given earlier with the following Latin square. Observe that each of the four indicated rectangles contains entries 1,2,3,4.*

1	2	4	5	8	9	7	3	6
3	4	2	1	5	7	8	6	9
9	5	1	2	4	8	6	7	3
2	1	3	4	9	6	5	8	7
5	3	7	6	1	2	4	9	8
8	7	6	9	3	4	2	1	5
7	8	9	3	6	5	1	2	4
6	9	5	8	7	1	3	4	2
4	6	8	7	2	3	9	5	1

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