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ASTRAL (n_4) CONFIGURATIONS OF PSEUDOLINES

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ABSTRACT. An *astral* (n_4) *configuration of pseudolines* is a collection in the Euclidean plane of n points and n pseudolines (that is, topological lines that have been modified in the Euclidean plane from straight lines only in a finite part, which cross each other only once, or have parallel infinite parts and are disjoint), where each point lies on four pseudolines, each pseudoline contains four points, and the points and pseudolines form two symmetry (transitivity) classes each. We describe how to construct astral (n_4) pseudoline configurations with dihedral symmetry, and we discuss the existence of astral (n_4) configurations with only chiral symmetry.

1. Introduction and Definitions

An (n_4) *configuration of points and lines* is a collection of points and lines in the Euclidean plane so that every point lies on four lines and every line passes through four points. (Such configurations are sometimes called *geometric* configurations, as opposed to combinatorial configurations.) Such a configuration is *astral* if there are precisely two symmetry (i.e., transitivity) classes of the points and the lines with respect to rotations and reflections of the plane mapping the configuration to itself. That is, an astral (n_4) configuration must have $n = 2m$ points and lines for some integer m.

The notion of configurations of points and lines may be generalized by replacing the straight lines by pseudolines to produce a *pseudoline configuration*, also known as a *topological configuration*. In the projective plane, a *pseudoline* is a simple closed curve that is topologically equivalent to a line (e.g., see [8, p. 40]). In the Euclidean plane, every pseudoline may be represented by a straight line that has been modified in a piecewise-linear fashion in a finite part so as to remain simple. A family of pseudolines has the additional restriction imposed that given any two pseudolines, either the infinite parts are parallel and the two pseudolines are disjoint, or the two pseudolines cross each other at a single point; that is, even though they may wiggle around somewhat, the pseudolines should "behave like lines". As in the linear case, we will say a pseudoline configuration is astral if there are

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two symmetry classes of pseudolines and two symmetry classes of points. An example of an astral (n_4) configuration of pseudolines is shown in Figure 1. A collection of pseudolines is said to be *stretchable* if it is isomorphic to a collection of straight lines; it is well-known that there exist non-stretchable arrangements of pseudolines [8].

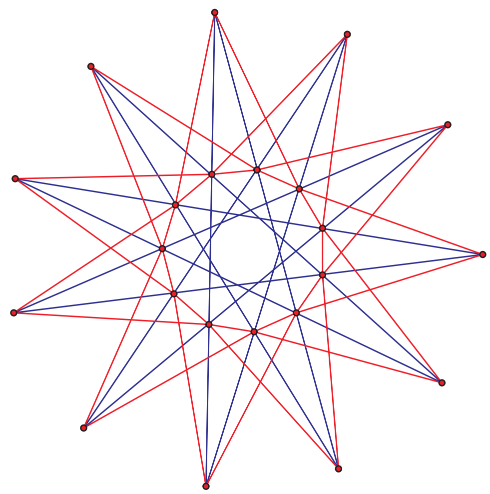


FIGURE 1. An astral (22_4) pseudoline configuration.

With the restriction of considering only straight lines, (n_4) configurations have been studied fairly extensively (see, e.g., [1, 3, 9, 10]); in [1, 9] it was established that *linear* astral (n_4) configurations exist if $n = 12m$ and $m > 1$. Addressing certain existence questions, Bokowski and Schewe [7] have investigated pseudoline (n_4) configurations and showed that there are no (n_4) pseudoline configurations with $n < 17$; Bokowski, Grünbaum and Schewe [6] showed that there exist (n_4) pseudoline configurations for every $n \geq 17$. However, no systematic investigation has been done on symmetric (that is, configurations with non-trivial geometric symmetry) (n_4) pseudoline configurations, although a few isolated examples of symmetric (in fact, astral) pseudoline (n_4) configurations have appeared in [6, 13]. In the remainder of the paper, we will discuss the existence and construction of (n_4) configurations that are astral, so that they have as much symmetry as possible.

Following [2], we define an astral type 1 (n_4) configuration to have each symmetry class of points form the vertices of a regular m-gon, and a type 2 astral (n_4) configuration to have each symmetry class of points form an isogonal but not regular m-gon (that is, the points are spaced long-short around a circle).

2. How small can they be?

Suppose we want to construct a type 1 astral $((2m)_4)$ configuration. By definition, every point has four pseudolines passing through it, and every pseudoline has four points lying on it, and the points may be partitioned into two symmetry classes, each containing m points. Note that the symmetry classes of points must form two concentric regular m-gons.

Label the points in the two symmetry classes as $v_0, v_1, \ldots, v_{m-1}$ and $w_0, w_1, \ldots, w_{m-1}$, and collectively call the sets $Z_m(v)$ and $Z_m(v)$, respectively. Without loss of generality, assume that the v_i are the outer ring of points and the w_i are the inner ring of points. Let $\mathcal{C}$ be the circumcircle of the regular m-gon formed by the points $Z_m(v)$. Define $((v_i, v_j))$ to be the number of vertices in $Z_m(v)$ between v_i and v_j, excluding v_i, including v_j, and travelling counterclockwise around $\mathcal{C}$; this is the *spanning distance* between the two points. A line or line segment that connects two vertices v_i and v_j where the spanning distance $((v_i, v_j)) = c$ (or similarly using elements of $Z_m(w)$) is said to be a line (segment) of *span* c. By convention, the span of a line (segment) is restricted to be at most $m/2$. Ideas for the proof of the following theorem were developed out of communications with Jürgen Bokowski.

Theorem 2.1. *For an astral $((2m)_4)$ configuration to exist, $m > 10$.*

Proof. Suppose an astral $((2m)_4)$ configuration exists. Consider point w_0. It lies on four pseudolines, two from each symmetry class. We will label the pseudolines as l_a and l'_a and l_b and l'_b, where l'_a is the image of l_a under rotation, and similarly with l_b. Since there should be two pseudolines of each symmetry class passing through each point, neither l_a nor l_b (or their images l'_a and l'_b) are diameters.

These pseudolines each must intersect the points $Z_m(v)$, say at points P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, and Q_4. Because the spans are distinct, we may assume that the endpoints of the pseudolines are assigned in such a way that in travelling counterclockwise around $\mathcal{C}$, the labels are ordered consecutively as P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4, so that l_a contains P_1 and Q_1, l_b contains P_2 and Q_2, l'_b contains P_3 and Q_3, and l'_a contains P_4 and Q_4 (see Figure 2). Note that since none of the pseudolines l_a, l'_a, l_b, l'_b are diameters, the distances $((P_i, Q_i)) \neq ((Q_i, P_i))$ for $i = 1, 2, 3, 4$.

By symmetry, $((P_1, Q_1)) = ((Q_4, P_4))$ and $((P_2, Q_2)) = ((Q_3, P_3))$, since l_a and l'_a (respectively, l_b and l'_b) are in the same symmetry class (that is, because l'_a is the rotation of l_a, but under rotation the labels switch, so that $P_1 \to Q_3$ and $Q_1 \to P_3$). Since l_a and l_b are *not* in the same symmetry class, $((P_1, Q_1)) \neq ((P_2, Q_2))$. Moreover, $((P_1, P_2)) \neq ((Q_1, Q_2))$ and $((P_3, P_4)) \neq ((Q_3, Q_4))$. To see this, note that if $((P_1, P_2)) = ((Q_1, Q_2))$,

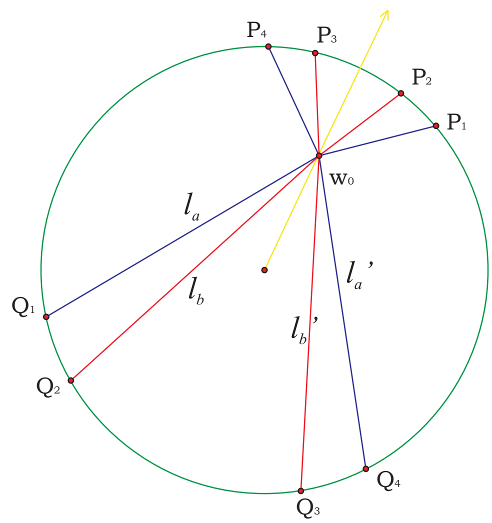


FIGURE 2. Illustration for the proof of Theorem 2.1.

then

$$\begin{aligned}((P_1, Q_1)) &= ((P_1, P_2)) + ((P_2, Q_1)) \\ &= ((P_2, Q_1)) + ((Q_1, Q_2)) \\ &= ((P_2, Q_2)),\end{aligned}$$

a contradiction, since l_a and l_b are assumed to be of different spans.

Finally, $((P_2, P_3)) \neq ((Q_2, Q_3))$. To see this, suppose not. Then

$$\begin{aligned}((P_2, Q_2)) &= ((P_2, P_3)) + ((P_3, P_4)) + ((P_4, Q_1)) + ((Q_1, Q_2)) \\ &= ((Q_2, Q_3)) + ((P_1, P_2)) + ((Q_4, P_1)) + ((Q_3, Q_4)),\end{aligned}$$

since $((P_2, P_3)) = ((Q_2, Q_3))$ by supposition, and $((P_4, Q_1)) = ((Q_4, P_1))$ and $((Q_1, Q_2)) = ((Q_3, Q_4))$ by symmetry.

But by rearranging terms, we see that

$$\begin{aligned}((P_2, Q_2)) &= ((Q_2, Q_3)) + ((P_1, P_2)) + ((Q_4, P_1)) + ((Q_3, Q_4)) \\ &= ((Q_2, Q_3)) + ((Q_3, Q_4)) + ((Q_4, P_1)) + ((P_1, P_2)) \\ &= ((Q_2, P_2))\end{aligned}$$

so l_b is a diameter, which is a contradiction.

Thus, for $i = 1, 2, 3$ we need at least one vertex between one of each of the pairs P_i, P_{i+1} or Q_i, Q_{i+1}, so the total number of vertices is at least $8 + 3 = 11$ (eight vertices are counted using P_i and Q_i, plus the three additional vertices). □

An (n_4) pseudoline configuration with 22 vertices ($m = 11$) is shown in Figure 1.

In [1], it was shown that linear astral (n_4) configurations exist if and only if $n = 12m$ and $m > 1$. The situation with pseudoline configurations is much less restrictive. We can generalize the configuration shown in Figure 1 to prove the following theorem; much more information about how to construct astral (n_4) pseudoline configurations will be given in the following sections.

Theorem 2.2. *For every $m > 10$, there exist astral $((2m)_4)$ pseudoline configurations.*

Proof. We will describe how to construct a $(2m_4)$ pseudoline configuration for any $m > 10$. Choose $m > 10$ and construct a regular m-gon; the vertices of this m-gon will be one of the symmetry classes of points of the $((2m)_4)$ configuration. On this m-gon, draw all the lines of span 5 (these are the blue lines in Figure 1). Then draw in span 4 pseudolines (shown in red in Figure 1) in such a way that these pseudolines pass through the intersections of the span 5 lines that are closest to the m-gon (the inner ring of points in Figure 1); these inner intersections form the second symmetry class of points of the configuration. By construction, the collection of span 5 lines, span 4 pseudolines and two classes of points forms an astral $((2m)_4)$ configuration of pseudolines. □

It will be sufficient to explicitly discuss only the construction of configurations whose symmetry classes of points form regular polygons (type 1 configurations); as in the linear case (see [1, Theorem 3]), if the points of an astral (n_4) pseudoline configuration form an isogonal but not regular m-gon (where m is divisible by 2), then the symmetry constraints of having only two classes of pseudolines force the connected component of a single vertex to be a type 1 configuration. That is,

Theorem 2.3. *If the vertices in each symmetry class of points in an astral (n_4) pseudoline configuration form concentric isogonal but non-regular polygons, then the pseudoline configuration is disconnected and is formed from two concentric copies of an (n_4) type 1 pseudoline configuration, where one copy is rotated through an arbitrary angle with respect to the other.*

3. Astral (n_4) configurations of pseudolines with dihedral symmetry

3.1. **Constructing configurations.** For the remainder of the paper, we will be discussing type 1 configurations, where the vertices in each symmetry class of points form regular m-gons. To construct type 1 astral (n_4) configurations of pseudolines with dihedral symmetry, we must begin with the vertices of a regular convex m-gon. Label these vertices v_i. We assume for convenience that

$$v_i = \left(\cos\left(\frac{2\pi i}{m}\right),\ \sin\left(\frac{2\pi i}{m}\right)\right).$$

The second ring of vertices w_i may be placed at an arbitrary radius. However, in order for the resulting configuration to have dihedral symmetry, the angle $\theta = \angle v_0 \mathcal{O} w_0$, where $\mathcal{O}$ is the center of the circumcircle of the v_i, must be an integer multiple of π/m. In particular, we assume that $\theta = 0$ or $\theta = \pi/m$.

In addition, type 1 astral (n_4) configurations have two symmetry classes of pseudolines, labelled $(l_a)_i$ and $(l_c)_i$. The configurations will be determined by seven parameters, which are listed in Table 1, and they will be denoted by $m\#\{(a, b; d, c), r, \theta\}$ for appropriate values of the parameters.

TABLE 1. Parameters determining an astral (n_4) configuration of pseudolines

m	the number of points in each symmetry class of points
a	the span formed on the outer ring of points by the pseudolines l_a
b	the span formed on the inner ring of points by the pseudolines l_a
c	the span formed on the outer ring of points by the pseudolines l_c
d	the span formed on the inner ring of points by the pseudolines l_c
r	the radius of the points labelled w_i
θ	the angle $\angle v_0 \mathcal{O} w_0$, the "offset"

We define the configuration as follows:

The points v_i are defined as above, and $\mathcal{O}$ is the origin. The point w_0 lies on the ray forming angle θ, measured counterclockwise, with the line $\langle \mathcal{O}, v_0 \rangle$ (that is, with the horizontal), at a distance of r from $\mathcal{O}$.

Pseudoline $(l_a)_0$ has an outer span of length a, so points v_0 and v_a lie on pseudoline $(l_a)_0$. It has an inner span of length b, so we need w_k and w_{k+b} to lie on $(l_a)_0$, for some choice of k.

To have a configuration with dihedral symmetry, we need each pseudoline to be in the shape of a "symmetric plateau": that is, the inner span segment $w_k w_{k+b}$ of the pseudoline must be parallel to the segment $v_0 v_a$. To achieve this, we need to count in from the bounding rays $\mathcal{O}v_0$ and $\mathcal{O}v_a$ by some fixed amount q (see Figure 3).

That is, the points w_i that form the endpoints of the inner span segment are w_q and w_{a-q} if $\theta = 0$ and w_q and w_{a-1-q} if $\theta = \pi/m$; by our choice of parameters, the span between them should be b. That is,

$$b = \begin{cases} (a-q) - q = a - 2q & \text{if } \theta = 0, \\ (a-1-q) - q = a - 2q - 1 & \text{if } \theta = \frac{\pi}{m}. \end{cases}$$

Since k (where w_k and w_{k+b} are the desired points on l_a) must be an integer, we let

$$k = \left\lfloor \frac{a-b}{2} \right\rfloor .$$

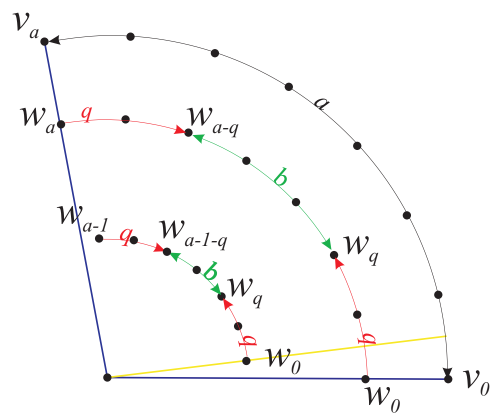


FIGURE 3. Determining indices for construction of the inner span. Here, the inner ring of points is the case where $\theta = \pi/m$ and the middle ring of points is the case where $\theta = 0$.

There are certain constraints on the parameters that are necessary for the resulting configuration to exist and have dihedral symmetry. For the remainder of the section, we will be developing these constraints.

Theorem 3.1. *Suppose $m\#\{(a, b; d, c), r, \theta\}$ is an astral (n_4) configuration of pseudolines with dihedral symmetry. If $\theta = 0$, then a and b (and c and d) must be of the same parity. If $\theta = \pi/m$, then a and b (and c and d) must be of opposite parity.*

There are some useful ways to rephrase this theorem. If $m\#\{(a, b; d, c), r, \theta\}$ is an astral (n_4) configuration of pseudolines with dihedral symmetry, then $\theta = [(a-b) \bmod 2]\cdot(\pi/m)$. Also, if $m\#\{(a, b; d, c), r, \theta\}$ is an astral (n_4) configuration of pseudolines with dihedral symmetry, then $a-b \equiv c-d \pmod 2$.

Proof. It suffices to prove this result for l_a and the points w_k and w_{k+b}. If $\theta = 0$, then the points w_0 and w_a actually lie on the rays $\overrightarrow{\mathcal{O}v_0}$ and $\overrightarrow{\mathcal{O}v_a}$, respectively, and are not available to be chosen as points on the inner span of pseudoline l_a. Therefore, the total angular distance from ray to ray is $a\pi/m$, so that $a = b + 2q$, and hence $a - b = 2q$, so a and b have the same parity.

On the other hand, if $\theta = \pi/m$, then all the points from w_0 to w_a are actually interior to the segment bounded by the rays $\overrightarrow{\mathcal{O}v_0}$ and $\overrightarrow{\mathcal{O}v_a}$, so that the total angular distance measured from each of the rays to the endpoint of the inner segment is actually $(q + \frac{1}{2})\frac{\pi}{m}$. Therefore, $a - 1 = b + 2q$, so $a - b = 2q + 1$ and a and b are of opposite parity. □

Lemma 3.2. *In a dihedrally symmetric astral (n_4) configuration of pseudolines with symbol $m\#\{(a, b; d, c), r, \theta\}$, neither a nor c may equal $m/2$.*

Proof. Suppose that $a = m/2$. Then the portion of the pseudoline $(l_a)_q$ that is outside the circumcircle of the v_i coincides with the portion of the pseudoline $(l_a)_{q+a}$ that is also outside the circumcircle, but they contain different inner points w_i, so the potential pseudolines l_a do not behave like pseudolines. □

By convention, we assume the parameters a, b, c, d are less than $m/2$. (The previous lemma and Theorem 3.4 show that they cannot be equal to $m/2$.)

"Traditional" pseudoline configurations have $b < a$ and $d < c$, and $0 < r < 1$. These are analogous to the linear astral (n_4) configurations, in that if m is a multiple of 6 and a, b, d, c are chosen appropriately, so that

$$\frac{\cos\left(\frac{a\pi}{m}\right)}{\cos\left(\frac{b\pi}{m}\right)} = \frac{\cos\left(\frac{c\pi}{m}\right)}{\cos\left(\frac{d\pi}{m}\right)} \quad \text{and} \quad r = \frac{\cos\left(\frac{a\pi}{m}\right)}{\cos\left(\frac{b\pi}{m}\right)},$$

then the resulting configuration is linear. Such configurations have been completely characterized; see Grünbaum [9] and Berman [1], with a discussion of the characterization (and a listing in the currently accepted notation) in [12]. An example of a linear configuration and the same configuration with a different radius is shown in Figure 4. Note that the notation we use here for astral (n_4) configurations of pseudolines corresponds to the notation used in [3], which was a slight modification of the notation introduced by Grünbaum in [12] (which itself was a modification of notation introduced by Boben and Pisanski in [5]) for a certain class of polycyclic (n_4) configurations which were named *celestial* configurations in [3]; linear astral configurations are a special class of these celestial configurations.

Explicitly, if m, a, b, d, c are valid parameters for a linear astral (n_4) configuration (say, as listed in [12]), then the configuration

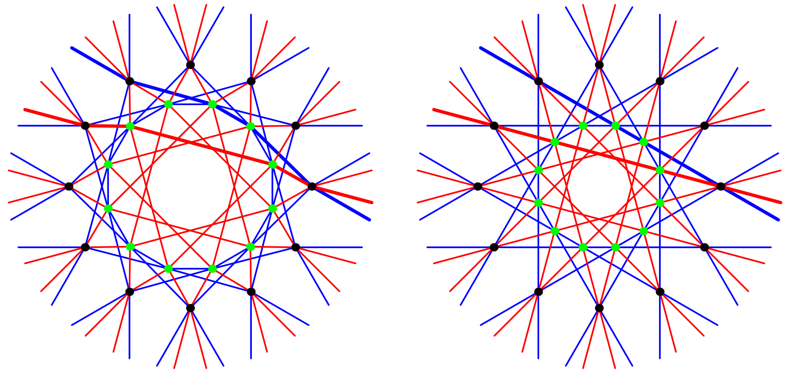
$$m\#\left\{(a, b; d, c), \frac{\cos\left(\frac{a\pi}{m}\right)}{\cos\left(\frac{b\pi}{m}\right)}, [(a+b) \bmod 2]\frac{\pi}{m}\right\}$$

is equivalent to the configuration $m\#(a, b; d, c)$ in the notation of [3] and $m\#(a, b, d, c)$ in the notation of [12], which in turn is equivalent to the configuration $m\#a_b\, c_d$ in the notation of [1, 2, 9]. Of course, since for linear astral (n_4) configurations to exist, $n = 12m$, not every set of parameters that produce a valid astral (n_4) pseudoline configuration corresponds to a linear configuration. In other words:

Theorem 3.3. *There exist infinitely many dihedrally symmetric astral* (n_4) *pseudoline configurations that are not stretchable.*

Proof. Since in [1] it was shown that every linear astral (n_4) configuration has $n = 12m$ and $m > 1$, every dihedrally symmetric astral (n_4) configuration of pseudolines $m\#\{(a, b; d, c), r, \theta\}$ with n not divisible by 12 is not stretchable. □

In particular, all the configurations constructed in Theorem 2.2 have dihedral symmetry, and those with m not divisible by 6 are not stretchable.



(a) (b)

FIGURE 4. A stretchable astral (n_4) pseudoline configuration. (a) The configuration $12\#\{(4,1;4,5), 0.7, \frac{\pi}{12}\}$; (b) The configuration $12\#\{(4,1;4,5), r, \frac{\pi}{12}\}$ with $r = \cos\left(\frac{4\pi}{12}\right) / \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{1+\sqrt{3}} \approx 0.517638$, which is the linear configuration $12\#(4,1;4,5)$ in the notation of [3].

3.2. **Constraints on the parameters caused by interactions within a single symmetry class.** Are there other ways to construct astral (n_4) pseudoline configurations with dihedral symmetry? Some constraints on the parameters in an astral (n_4) configuration of psuedolines are caused by needing pseudolines in a single class to intersect appropriately; that is, pseudolines in a single class may intersect other pseudolines of that class at most once.

Theorem 3.4. *Parameters for a dihedrally symmetric astral (n_4) pseudoline configuration with symbol $m\#\{(a,b;d,c), r, \theta\}$ satisfy $b < a$ and $d < c$.*

Proof. It suffices to show that $b < a$. Suppose not. That is, suppose that $b \geq a$. We will show that two pseudolines in the potential pseudoline class l_a intersect each other at least twice (see Figure 5).

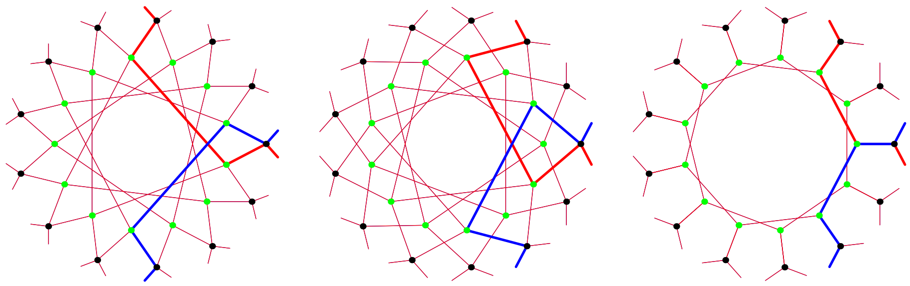
By construction, $(l_a)_j$ contains the points v_j, w_{k+j}, w_{k+b+j} and v_{a+j}. Therefore, pseudoline $(l_a)_0$ contains points v_0, w_k, w_{k+b} and v_a, and pseudoline $(l_a)_{-a}$ (with indices understood modulo m) contains points v_{-a}, w_{k-a}, w_{k+b-a} and v_0.

Since $b \geq a$,

$$k = \left\lfloor \frac{a-b}{2} \right\rfloor \leq 0$$

and

$$k + b - a = \left\lfloor \frac{a-b}{2} \right\rfloor + b - a \geq 0.$$



(a) (b) (c)

FIGURE 5. If $b \geq a$, the pseudolines l_a intersect too many times; in each case, pseudoline $(l_a)_0$ is shown in thick red and $(l_a)_{-a}$ is shown in thick blue. (a) $m = 13$, $a = 3$, $b = 4$, $\theta = \frac{\pi}{13}$; (b) $m = 13$, $a = 2$, $b = 4$, $\theta = 0$; (c) $m = 13$, $a = 2$, $b = 2$, $\theta = 0$.

If $b = a$ (and therefore $\theta = 0$), then the two potential pseudolines $(l_a)_0$ and $(l_a)_{-a}$ share points v_0 and w_0 and hence the entire segment v_0w_0 and therefore clearly do not qualify as pseudolines.

If $b > a$, then travelling along pseudoline $(l_a)_0$, beginning at v_0 and only considering angle, you travel clockwise, crossing the ray $\overrightarrow{\mathcal{O}v_0}$, to reach w_k and then counterclockwise, recrossing the ray to reach w_{k+b}. However, travelling along pseudoline $(l_a)_{-a}$, requires travelling counterclockwise from w_{k-a} to w_{k+b-a}, crossing the ray $\overrightarrow{\mathcal{O}v_0}$, and then clockwise, again crossing the ray, to reach v_0. By continuity, these two pseudolines must therefore cross each other at least once between w_{k-a} and v_0 for pseudoline $(l_a)_{-a}$ and between v_0 and w_k for pseudoline $(l_a)_0$, and the two pseudolines also touch at v_0. Therefore, the pseudolines $(l_a)_{-a}$ and $(l_a)_0$, which are both members of class l_a, intersect each other twice and thus do not qualify as pseudolines.

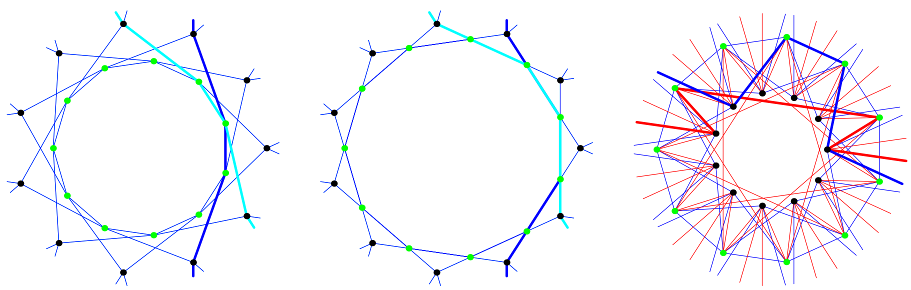
□

Proposition 3.5. *In an astral (n_4) configuration of pseudolines with symbol $m\#\{(a, b; d, c), r, \theta\}$, there are constraints on r depending on the values of a, b, c, d.*

The precise dependence of r on the other parameters seems to be hard to determine explicitly. However, clearly, if r is too large (say, significantly larger than 1; see Figure 6(c)), then the potential pseudolines intersect each other more than once. An example of the potential problems is shown in Figure 6. We propose the following:

Conjecture. *In a configuration* $m\#\{(a,b;d,c),r,\theta\}$ *with* $\theta = [a+b \bmod 2]\frac{\pi}{m}$,

$$0 < r < \frac{\cos\left(\frac{(a-b-1)\pi}{m}\right)}{\cos\left(\frac{\pi}{m}\right)}.$$



(a) (b) (c)

Figure 6. Constraints on the radius are needed. Figures (a) and (b) consist of pseudolines l_a with $m = 11$, $a = 4$, $b = 1$, and $\theta = \pi/11$, with two pseudolines in the symmetry class (which intersect badly in the left-hand image) shown thick, one in blue and one in cyan. (a) $r = 0.7$; (b) $r = \cos\left(\frac{(4-1-1)\pi}{11}\right)\Big/\cos\left(\frac{\pi}{11}\right) \approx 0.876769$. (c) As r gets big, pseudolines intersect badly. This has two symmetry classes of pseudolines (the configuration shown in Figure 1, only with $r = 2$), with symbol $11\#\left\{(4,1;4,5),2,\frac{\pi}{11}\right\}$, and very bad intersections.

Figure 6(b) shows an example of the intersections between pseudolines in a single symmetry class that result when r is chosen to equal the upper bound given in the conjecture; Figure 6(a) shows the same symmetry class of pseudolines when r is in the range allowed by the conjecture.

3.3. **Constraints on parameters caused by interactions between the two symmetry classes of pseudolines.** Some constraints on the parameters for an astral (n_4) configuration of pseudolines are caused by the interaction between the two symmetry classes of pseudolines; that is, certain choices of parameters cause (proposed) pseudolines in the two classes to intersect inappropriately.

Lemma 3.6. *In a dihedrally symmetric astral* (n_4) *pseudoline configuration* $m\#\{(a,b;d,c),r,\theta\}$, $b \neq d$.

Proof. This follows straightforwardly from the assignment of points to lie on pseudolines of class l_a and l_c. If $b = d$, then pseudolines of class l_a and l_c will both contain inner spans b (that is, they will contain points of the form

w_j and w_{j+b} for some value of j, which will be different for pseudolines in each class) and so will coincide on the inner spans. ☐

Lemma 3.7. *In a dihedrally symmetric astral* (n_4) *pseudoline configuration* $m\#\{(a,b;d,c),r,\theta\}$*,* $a-b \neq c-d$*.*

Proof. This follows immediately from the definition of k for each of the pseudolines l_a and l_c. If $a-b=c-d=x$, then pseudoline $(l_a)_0$ contains points v_0 and $w_{\lfloor x/2 \rfloor}$ and pseudoline $(l_c)_0$ also contains points v_0 and $w_{\lfloor x/2 \rfloor}$. Therefore, the two potential pseudolines do not actually qualify as pseudolines. ☐

In fact, we can prove an even stronger result.

Theorem 3.8. *Suppose that* $m\#\{(a,b;d,c),r,\theta\}$ *is a dihedrally symmetric astral pseudoline* (n_4) *configuration and* $a<c$*; then* $a-b>c-d$*.*

Proof. Suppose not; that is, suppose $a-b \leq c-d$. By Lemma 3.7, in fact, $a-b<c-d$. By construction, both pseudolines $(l_a)_0$ and $(l_c)_0$ intersect at v_0. The pseudoline $(l_a)_0$ contains point v_a and $(l_c)_0$ contains v_c, and by hypothesis, $a<c$, so the angle $\angle v_0\mathcal{O}v_a < \angle v_0\mathcal{O}v_c$. That is, on the circumcircle of $Z_m(v)$, travelling counterclockwise, v_a comes before v_c.

If by our supposition $a-b<c-d$, then $k \leq j$, where $j=\lfloor \frac{c-d}{2} \rfloor$ and $k=\lfloor \frac{a-b}{2} \rfloor$. In fact, since by Theorem 3.1, $a-b \equiv c-d \bmod 2$, if $a-b<c-d$ then $k<j$.

It is convenient to consider a few cases relating the positions of the points in $Z_m(w)$ on the pseudolines $(l_a)_0$ and $(l_c)_0$: (1) w_{k+b} is between w_j and w_{j+c}, shown in Figure 7(a); (2) w_j and w_{j+c} are both between w_k and w_{k+b}; (3) w_j occurs after w_{k+b}, shown in Figure 7(b).

Case 1: $j \leq k+b$:

In this case, in counterclockwise order around the circumcircle of $Z_m(w)$, we have w_k, w_j, w_{k+b} where the first and last points lie on $(l_a)_0$ and the middle on $(l_c)_0$. In this case, the line segment w_kw_{k+b} crosses the line segment w_jw_{j+c}, so the potential pseudolines $(l_a)_0$ and $(l_c)_0$ intersect twice.

Case 2: $j<j+c<k+b$:

If this were possible, then the order of the points in $Z_m(w)$ on pseudolines $(l_a)_0$ and $(l_c)_0$ would be $w_k, w_j, w_{j+c}, w_{k+b}$. However, by Theorem 3.4 we have that $b<a$ and by hypothesis $a<c$ and $k<j$, so

$$k+b<j+b<j+c$$

contradicting our case assumption that $j+c<k+b$. Thus this case is impossible.

Case 3: $j>k+b$:

Here, the counterclockwise order of the points in $Z_m(w)$ on pseudolines $(l_a)_0$ and $(l_c)_0$ is w_k, w_{k+b}, w_j, w_{j+c}; in this case, because $a<c$ and hence v_a is before v_c, the segment v_0w_j crosses the segment $w_{k+b}v_a$,

contradicting our assumption that $(l_a)_0$ and $(l_c)_0$ intersect only once, at v_0.

$\square$

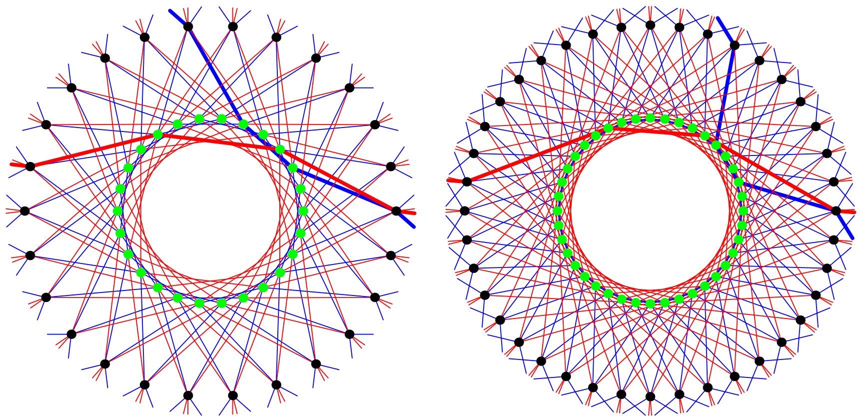


FIGURE 7. Illustrations of the cases in Theorem 3.8. (a) $j \leq k+b$ (symbol $26\#\left\{(7,3;6,12),\frac{1}{2},0\right\}$); (b) $j > k+b$ (symbol $40\#\left\{(7,3;7,19),\frac{1}{2},0\right\}$)

In conclusion, the only possible astral (n_4) configurations with dihedral symmetry with symbols $m\#\{(a,b;d,c),r,\theta\}$ are the "traditional" configurations which have $b < a$, $d < c$, θ equal to 0 or π/m, and small enough r; moreover, if we assume that $a < c$, we must have that $a-b > c-d$. Of course, $a > c$ is possible; this corresponds to switching the role of the two symmetry classes of pseudolines. (That is, the configurations $m\#\{(a,b;d,c),r,\theta\}$ and $m\#\{(c,d;b,a),r,\theta\}$ are isomorphic, but the red and blue pseudolines will be switched.)

4. ASTRAL (n_4) CONFIGURATIONS WITH CHIRAL SYMMETRY

Unlike in the linear case, there are lots of astral configurations of pseudolines with chiral (that is, only rotational) symmetry! A small example is shown in Figure 8.

Theorem 4.1. *If $m\#\{(a,b;c,d),r,\theta\}$ is a valid configuration symbol for a dihedrally symmetric configuration, then using the same values of m,a,b,d,c,r where $\theta \neq [(a-b) \bmod 2]\cdot\frac{\pi}{m}$ but is sufficiently close to it results in a configuration with chiral symmetry.*

Figure 8 shows the configuration $11\#\{(4,1;4,5),0.4,0\}$, which is isotopic to the dihedrally symmetric configuration $11\#\left\{(4,1;4,5),0.4,\frac{\pi}{11}\right\}$ shown in Figure 1.

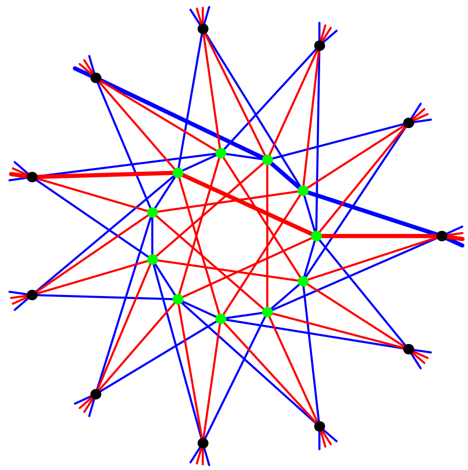


FIGURE 8. A chirally symmetric astral (22_4) configuration of pseudolines.

Proof. These are just the dihedrally symmetric configurations, with the angle between v_0 and w_0 rotated slightly. If θ is too far away from $[(a-b) \bmod 2] \cdot \frac{\pi}{m}$, then pseudolines in the classes l_a and l_c may touch or cross themselves or each other because of too much twisting. □

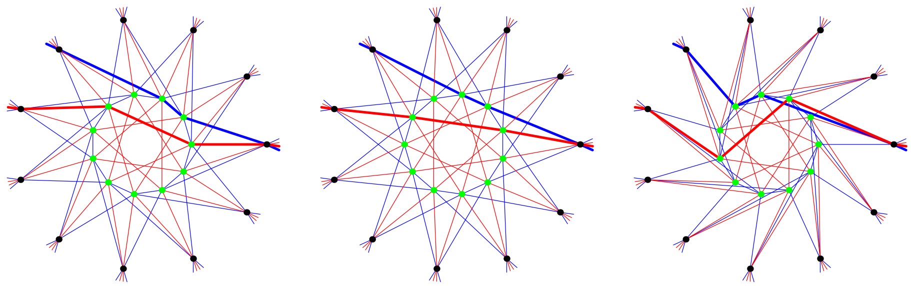
The theorem may be rephrased as follows.

Corollary 4.2. *There exist continuous families of chirally symmetric astral (n_4) configurations of pseudolines which are all isotopic to a dihedrally symmetric configuration.*

It is unclear what the precise constraints on θ should be so that the pseudolines have no undesirable intersections; an example of a chiral pseudoline configuration that has been twisted too far is shown in Figure 9, along with a chiral pseudoline configuration and the corresponding dihedral configuration.

There is more to the story, however: there are many other chirally symmetric astral (n_4) configurations of pseudolines which are *not* isomorphic to any dihedral configuration.

As an easy example, suppose that we wished to construct a configuration where $m = 20$, which had pseudolines l_a containing points $v_j, w_{j+2}, w_{j+4}, v_{j+6}$ and pseudolines l_c containing points $v_j, w_j, w_{j+8}, v_{j+9}$. If we construct this configuration with $\theta = 0$, shown in Figure 10(a), the pseudolines intersect appropriately and the configuration is an astral (n_4) configuration of pseudolines with chiral symmetry. We can observe that the outer spanning distance of the pseudolines l_a is 6, the inner spanning distance (on the w_i) of the pseudolines l_a is 2, the outer spanning distance on the pseudolines l_c is 9 and the inner spanning distance on the pseudolines l_c is 8. (That is, the lines l_a have $a = 6$ and $b = 2$, and the lines l_c have $c = 9$ and $d = 8$.)



(a) $\theta = 0$ (b) $\theta = \frac{\pi}{11}$ (c) $\theta = \frac{4\pi}{11}$

FIGURE 9. Collections of points and pseudolines corresponding to the symbol $11\#\{(4,1;4,5), 0.4, \theta\}$. The two leftmost pictures are astral pseudoline configurations; (a) has chiral symmetry and (b) has dihedral symmetry. However, in (c), where $\theta = 4\pi/11$, the potential pseudolines intersect inappropriately; for example, the thick red pseudoline intersects the thick blue pseudoline twice.

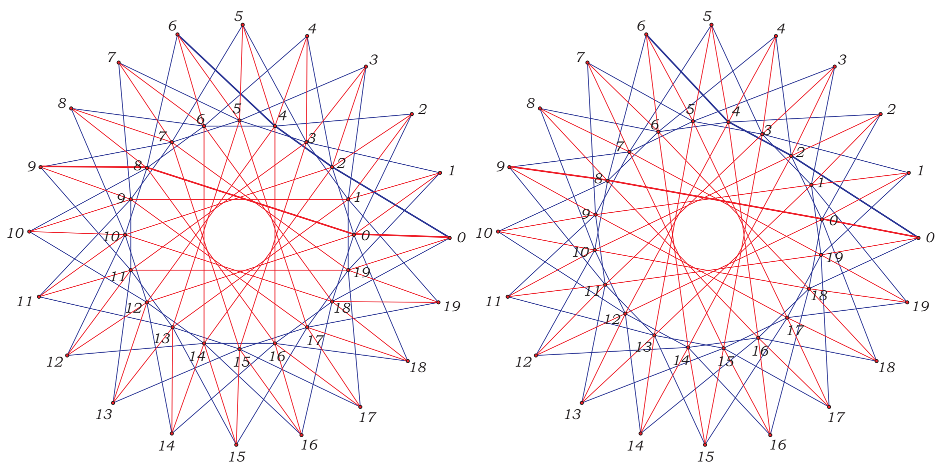


FIGURE 10. A chiral configuration which is not equivalent to any dihedral configuration. (a) The chiral configuration with $\theta = 0$; the lines l_a have dihedral symmetry, but the lines l_c do not. (b) The chiral configuration with $\theta = \pi/m$; the lines l_c have dihedral symmetry, and the lines l_a do not.

By the same reasoning used to prove Theorem 3.1, since the outer and inner spanning distances of lines l_a are of the same parity, the lines l_a will have dihedral symmetry when $\theta = 0$. However, since the outer and inner spanning distances of lines l_c are of the opposite parity, these lines will have dihedral symmetry when $\theta = \pi/m$, shown in Figure 10(b). In either case, the other set of lines will not have dihedral symmetry. Therefore, the entire configuration will have only chiral symmetry for any choice of θ.

Because of the symmetry constraints, all dihedrally symmetric astral (n_4) pseudoline configurations may be represented as discussed above, where each pseudoline consists of three line segments and two rays.

Question. *Does there exist a chirally symmetric astral* (n_4) *pseudoline configuration that can not be represented using pseudolines formed from three line segments and two rays?*

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