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ON PRIMITIVE SYMMETRIC ASSOCIATION SCHEMES WITH $m_1 = 3$

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ABSTRACT. We classify primitive symmetric association schemes with $m_1 = 3$. Namely, it is shown that the tetrahedron, i.e., the association scheme of the complete graph K_4, is the unique such association scheme. Our proof of this result is based on the spherical embeddings of association schemes and elementary three dimensional Euclidean geometry.

1. INTRODUCTION.

Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be a symmetric association scheme. That is, $\mathfrak{X}$ is a pair of a finite set X with cardinality $|X| = n$ and a set of relations $R_i (0 \le i \le d)$ satisfying certain conditions. The reader is referred to [2] and/or [3] for the definition and the basic properties of association schemes.

Let $A_i (0 \le i \le d)$ be the adjacency matrix with repect to the relation $R_i (0 \le i \le d)$ on X, and let $\mathfrak{A} = \langle A_0, A_1, \ldots, A_d \rangle$ be the Bose-Mesner algebra of $\mathfrak{X}$. Let $E_i (0 \le i \le d)$ be the primitive idempotents of $\mathfrak{A}$. We denote the eigenmatrices of $\mathfrak{X}$ by P and Q. Let $k_i (= p_{ii}^0 = P_i(0))$ be the subdegrees of $\mathfrak{X}$, and let $m_i (= q_{ii}^0 = Q_i(0) =$ rank of $E_i)$ be the dual subdegrees of $\mathfrak{X}$.

The purpose of this paper is to prove the following theorem.

Theorem 1. *Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be a primitive symmetric association scheme with $m_1 = 3$. Then $\mathfrak{X}$ must be the association scheme of tetrahedron, i.e., the association scheme with $d = 1$ and $|X| = 4$.*

In the rest of this Introduction, we give a brief sketch of the proof of Theorem 1. Let $\mathfrak{X}$ be an association scheme satisfying the assumptions of Theorem 1. Then X can be embedded in the unit sphere in the real Euclidean space $\mathbb{R}^3$ in such a way that two elements x and y of X (in the unit sphere S^2) in the relation R_i have the fixed inner product $\frac{1}{3}Q_1(i)$. By renumbering the relations $R_1, R_2, \ldots, R_d$ if necessary, we may assume without loss of generality that $\frac{1}{3}Q_1(1) \ge \frac{1}{3}Q_1(i)$ for all i with $1 \le i \le d$. By using the

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spherical embedding of the association scheme $\mathfrak{X}$ in S^2, we can conclude that $k_1 \leq 5$. Again, using the spherical representation of the association scheme, we prove that the cases $k_1 = 5$, 4 and 3 are impossible. This will be shown in Sections 4, 5 and 6, respectively, and we complete the proof of Theorem 1.

The result proved in this paper was originally obtained by the first author, and a preprint was circulated in the preprint series of Kyushu University (KYUSHU-MPS-1996-3, June 1996). The result was also announced in the Workshop on Distance Regular Graphs organized by G. Hahn and G. Sabidussi held in Montreal in Nov. 1996. In that preprint, the classification of quasi-regular polyhedrons (cf. [4], [6] and [7]) and the classification of primitive symmetric association schemes with $k_1 = 3$ by Yamazaki [8] were used. Subsequently, with the help of the second author, the paper was revised by adapting more elementary approach and by improving the exposition of the proofs. This revised version was included in our book [1] (written in Japanese) in 1999. The content of the present paper is essentially an English version of it with some improvements.

2. Basic facts

Definition 2. *An Association scheme* $\mathfrak{X} = (X, \{R_i\}_{0\leq i\leq d})$ *is* **imprimitive** *if there exists a nonempty proper subset* Λ $(\neq \{0\})$ *of* $\{0, 1, \ldots, d\}$ *for which* $\cup_{i\in\Lambda} R_i$ *defines an equivalence relation on the set* X. *If* $\mathfrak{X}$ *is not imprimitive, then* $\mathfrak{X}$ *is called* **primitive**.

Let $\Gamma_i = (X, R_i)$ be the graph defined on X with R_i for some i. For $\mathbf{x}$, $\mathbf{y} \in X$, define $\mathbf{x} \sim_i \mathbf{y}$ if there exists a path from $\mathbf{x}$ to $\mathbf{y}$ in the graph Γ_i. This $\sim_i$ gives an equivalence relation on X (we consider $\mathbf{x} = \mathbf{x}_0 = \mathbf{x}$ as a path from $\mathbf{x}$ to $\mathbf{x}$ of length 0). Let

$$A_i{}^l = \sum_{j=0}^{d} \alpha_{i,l,j} A_j$$

and define $\Lambda_i \subseteq \{0, 1, \ldots, d\}$ by

$$\Lambda_i = \{j \in \{0, 1, \ldots, d\} \mid \alpha_{i,l,j} \neq 0 \text{ for some } l\}.$$

Then we have the following well known proposition.

Proposition 3. $\mathbf{x} \sim_i \mathbf{y}$ *if and only if* $(\mathbf{x}, \mathbf{y}) \in R_j$ *with some* $j \in \Lambda_i$.

Proposition 3 implies that $\cup_{j\in\Lambda_i} R_j$ defines an equivalence relation on X. The following proposition is also well known, however we will give a proof of it for the reader's convenience.

Proposition 4. *Let* $\mathfrak{X}$ *be a primitive symmetric association scheme. Then the following (1), (2) and (3) hold.*

(1) *For any i, with $1 \leq i \leq d$, the graph Γ_i is connected.*
(2) $k_i \geq 2$ *holds for any $i \neq 0$.*
(3) *If $k_i = 2$ for some i, then $k_j = 2$ holds for $j \neq 0$, $|X|$ is a prime number, and $\mathfrak{X}$ is the association scheme of the regular p-gon with p a prime mumber.*

Proof Proposition 3 implies (1) and (2) immediately. We will prove (3). Since the graph Γ_i is connected, if $k_i = 2$, the graph is an n-gon. Without loss of generarity, we may assume $k_1 = 2$. Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and $(\mathbf{x}_1, \mathbf{x}_2), (\mathbf{x}_2, \mathbf{x}_3), \ldots, (\mathbf{x}_{n-1}, \mathbf{x}_n), (\mathbf{x}_n, \mathbf{x}_1) \in R_1$. If $3 \leq [\frac{n}{2}]$, then $(\mathbf{x}_1, \mathbf{x}_3) \notin R_1$. We may assume $(\mathbf{x}_1, \mathbf{x}_3) \in R_2$ by reordering the relations $R_2, \ldots, R_d$ if necessary. This implies $p^1_{2,1} = 1$ and $p^2_{1,1} \neq 0$. Therefore $(\mathbf{x}, \mathbf{y}) \in R_2$ if and only if there is a path $\mathbf{x} \sim_1 \mathbf{y}$ of length 2. Since Γ_1 is an n-gon, $k_2 = 2$ holds. If $4 \leq [\frac{n}{2}]$, then $(\mathbf{x}_1, \mathbf{x}_4) \notin R_1 \cup R_2$. We may assume $(\mathbf{x}_1, \mathbf{x}_4) \in R_3$. By a similar argument as given above we can show $(\mathbf{x}, \mathbf{y}) \in R_3$ if and only if there is a path $\mathbf{x} \sim_1 \mathbf{y}$ of length 3 and $k_3 = 2$. If we continue this process, then we will find out that $(\mathbf{x}, \mathbf{y}) \in R_j$ if and only if there exists a path $\mathbf{x} \sim_1 \mathbf{y}$ of length j and $k_j = 2$ for any $j \leq [\frac{n}{2}]$. Since the graph Γ_j is connected for any j, the cardinality n of X cannot be a multiple of any j with $2 \leq j \leq [\frac{n}{2}]$. Hence n must be a prime number p and $\mathfrak{X}$ is the association scheme of a regular p-gon. □

Next we will explain how to embed X in a unit sphere in Euclidean space. We have

$$E_1 = \frac{1}{|X|} \sum_{j=0}^{d} Q_1(j) A_j. \tag{2.1}$$

Since $\mathfrak{X}$ is symmetric, all the entries of the eigen matrices P and Q of $\mathfrak{X}$ are real numbers. If necessary, change the ordering of $A_1 \ \ldots, \ A_d$ and we may assume $Q_1(1) \geq Q_1(j)$ for any $1 \leq j \leq d$. Let V be the vector space over the real number field $\mathbb{R}$ indexed by the set X. Let $\{\mathbf{e}_{\mathbf{x}}, \ \mathbf{x} \in X\}$ be the canonical basis of V. Let $V_1 = VE_1$. Then $\dim(V_1) = \operatorname{rank}(E_1) = m_1$. Let $\overline{\mathbf{x}} = \sqrt{\frac{n}{m_1}} \mathbf{e}_{\mathbf{x}} E_1 \in V_1$. Let us denote the inner product between 2 vectors $\overline{\mathbf{x}}, \overline{\mathbf{y}}$ by $\overline{\mathbf{x}} \cdot \overline{\mathbf{y}}$. Then we have

$$\overline{\mathbf{x}} \cdot \overline{\mathbf{y}} = \frac{n}{m_1} \mathbf{e}_{\mathbf{x}} E_1 \, {}^t(\mathbf{e}_{\mathbf{y}} E_1) = \frac{n}{m_1} e_{\mathbf{x}} E_1 \, {}^t(\mathbf{e}_{\mathbf{y}}) = \frac{n}{m_1} E_1(\mathbf{x}, \mathbf{y}).$$

Then equation (2.1) implies

$$\|\overline{\mathbf{x}}\|^2 = \overline{\mathbf{x}} \cdot \overline{\mathbf{x}} = \frac{n}{m_1} \frac{Q_1(0)}{n} = 1.$$

Hence $\{\overline{\mathbf{x}} \mid \mathbf{x} \in X\}$ is on the unit sphere of the m_1-dimensional Euclidean space V_1. Next we prove that $\overline{\mathbf{x}} = \overline{\mathbf{y}}$ if and only if $\mathbf{x} = \mathbf{y}$. If $(\mathbf{x}, \mathbf{y}) \in R_j$ with $j \neq 0$ and $\overline{\mathbf{x}} = \overline{\mathbf{y}}$, then $1 = \overline{\mathbf{x}} \cdot \overline{\mathbf{y}} = \frac{n}{m_1} E_1(\mathbf{x}, \mathbf{y})$. Then equation (2.1) implies

$$Q_1(j) = m_1.$$

Let $\Lambda = \{i \mid Q_1(i) = m_1\}$. The matrix Q is nonsingular and $Q_0(i) = 1$ for any i, $0 \le i \le d$. Hence Λ has to be a proper subset of $\{0, 1, \dots, d\}$. It is easy to see that $\cup_{i\in\Lambda} R_i$ gives an equivalence relation on X. This contradicts the primitivity of $\mathfrak{X}$. Therefore $Q_1(j) = m_1$ holds if and only if $j = 0$. Thus we have seen that $\mathbf{x} \longrightarrow \overline{\mathbf{x}}$ gives a one to one correspondence between X and the subset $\overline{X} = \{\overline{\mathbf{x}} \mid \mathbf{x} \in X\} \subset S^{m_1-1} \subset V_1 \cong \mathbb{R}^{m_1}$.

Remark If $\mathfrak{X}$ is a primitive symmetric association scheme and $m_1 = 2$, then $\overline{X}$ is on the unit circle in $\mathbb{R}^2$. Therefore $k_i = 2$ holds for any i, $1 \le i \le d$ and $\mathfrak{X}$ is a regular p-gon with p a prime number.

3. $k_1 \le 5$

In this section we prove that $k_1 \le 5$, under the assumptions of Theorem 1. Therefore $m_1 = 3$ and $\overline{X}$ is a subset of S^2. In the following we identify X and $\overline{X}$. Let $A(X) = \{\mathbf{x}\cdot\mathbf{y} \mid \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \ne \mathbf{y}\}$ $(\subset \mathbb{R})$. Then $A(X) = \{\frac{Q_1(j)}{3} \mid 1 \le j \le d\}$. For i with $1 \le i \le d$, $\alpha \in A(X)$ and $\mathbf{x} \in X$, let $R_i(\mathbf{x}) = \{\mathbf{y} \in X \mid (\mathbf{x}, \mathbf{y}) \in R_i\}$ and $\Gamma_\alpha(\mathbf{x}) = \{\mathbf{y} \in X \mid \mathbf{x}\cdot\mathbf{y} = \alpha\}$. Let $\Lambda_\alpha = \{i \mid \frac{Q_1(i)}{3} = \alpha\}$ for any $\alpha \in A(X)$. Then by definition we have $k_i = |R_i(\mathbf{x})|$ for any i, $0 \le i \le d$. We will prove the following proposition.

Proposition 5. (1) *If $R_i(\mathbf{x}) \cap \Gamma_\alpha(\mathbf{x}) \ne \emptyset$, then $R_i(\mathbf{x}) \subseteq \Gamma_\alpha(\mathbf{x})$.*

(2) *Let Γ_α be the graph defined on X by $\{(\mathbf{x}, \mathbf{y}) \in X \times X \mid \mathbf{x}\cdot\mathbf{y} = \alpha\}$. Then Γ_α is regular.*

(3) *Let $\alpha = \frac{Q_1(1)}{3}$, the maximum real number in $A(X)$. Then $k_1 = |R_1(\mathbf{x})| = |\Gamma_\alpha(\mathbf{x})| \le 5$.*

Proof (1) If $(\mathbf{x}, \mathbf{y}) \in R_i(\mathbf{x}) \cap \Gamma_\alpha(\mathbf{x})$, then $\alpha = \frac{Q_1(i)}{3}$. Then $\mathbf{x}\cdot\mathbf{z} = \frac{Q_1(i)}{3} = \alpha$ holds for any $\mathbf{z} \in R_i(\mathbf{x})$.

(2) (1) implies that $\Gamma_\alpha(\mathbf{x}) = \cup_{i\in\Lambda_\alpha} R_i(\mathbf{x})$ holds. Hence $|\Gamma_\alpha(\mathbf{x})| = \sum_{i\in\Lambda_\alpha} k_i$ holds and the graph Γ_α is regular, and it's valency is $\sum_{i\in\Lambda_\alpha} k_i$.

(3) Since α is the maximum real number in $A(X)$, points $\mathbf{x}, \mathbf{y} \in X$ with $\mathbf{x}\cdot\mathbf{y} = \alpha$ gives the minimum distance $a = \sqrt{2(1-\alpha)}$ between the distinct points in X. Without loss of generality we may assume $\mathbf{x} = (0, 0, 1)$. First we will show that $|\Gamma_\alpha(\mathbf{x})| \le 5$ holds. Let $S = \{\mathbf{y} \in S^2 \mid \mathbf{x}\cdot\mathbf{y} = \alpha\}$. Then S is a circle on the plane $\{(x, y, z) \in \mathbb{R}^3 \mid z = \alpha\}$ whose center is $(0, 0, \alpha)$ and radius $\sqrt{1-\alpha^2}$ (see *fig*.1 given above). Since $-1 \le \alpha < 1$, we have $\sqrt{1-\alpha^2} < \sqrt{2(1-\alpha)}$. This means the radius of S is strictly less than the minimum distance $\sqrt{2(1-\alpha)}$ of the points in X. Since the kissing number of the unit cirles in $\mathbf{R}^2$ is 6, $\Gamma_\alpha(\mathbf{x})$ contains at most 5 points. Next, we note that, from the primitivity of $\mathfrak{X}$, and Proposition 4, each $k_i (i \ge 1)$ must be at least 2, and if $k_j = 2$ with some j, then $\mathfrak{X}$ is the association scheme of a regular p-gon for some prime number p. If X is the association scheme of a

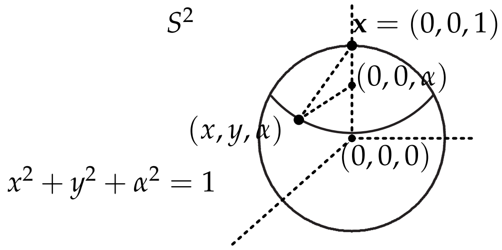

FIGURE 1

regular p-gon, then $m_i \le 2$ holds for any i, which contradicts the assumption $m_1 = 3$. Hence Proposition 5 implies $3 \le k_1 = |R_1(\mathbf{x})| \le |\Gamma_\alpha(\mathbf{x})| \le 5$. Since $|\Gamma_\alpha(\mathbf{x})| = \sum_{i\in\Lambda_\alpha} k_i$ and $k_i \ge 3$, we must have $k_1 = |\Gamma_\alpha(\mathbf{x})| \le 5$. Hence $R_1(\mathbf{x}) = \Gamma_\alpha(\mathbf{x})$ holds. □

Proposition 6. *Let $\alpha = \frac{Q_1(1)}{3}$, the maximum real number in $A(X)$. Then the graph defined on $\Gamma_\alpha(\mathbf{x})$ by $\{(\mathbf{u},\mathbf{v}) \in \Gamma_\alpha(\mathbf{x}) \mid \mathbf{u}\cdot\mathbf{v} = \beta\}$ is regular for any $\mathbf{x} \in X$ and $\beta \in A(X)$.*

Proof Proposition 5 tells us that $\Gamma_\alpha(\mathbf{x}) = R_1(\mathbf{x})$ for any $\mathbf{x} \in X$. Let $\mathbf{u} \in \Gamma_\alpha(\mathbf{x})$. Then Proposition 5 implies that

$$|\{\mathbf{v} \in \Gamma_\alpha(\mathbf{x}) \mid \mathbf{v}\cdot\mathbf{u} = \beta\}| = \sum_{j\in\Lambda_\beta} p^1_{j,1},$$

where $\Lambda_\beta = \{j \mid \frac{Q_1(j)}{3} = \beta\}$. □

Remark. The argument in this section shows in general that if a primitive symmetric (or commutative) association scheme has a given $m_1 = m$, then at least one of the $k_i (1 \le i \le d)$ is bounded by a function depending only on m.

4. THE IMPOSSIBILITY OF THE CASE $k_1 = 5$

Let us assume that $k_1 = 5$. Let $\mathbf{x}_0 \in X(\subset S^2)$. We note that $\alpha = \frac{1}{3}Q_1(1)$ is the maximum value among the real numbers in $A(X)$. Then $\Gamma_\alpha(\mathbf{x}_0) = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_5\}$ is on the circle $S = \{\mathbf{y} \in S^2 \mid \mathbf{x}_0\cdot\mathbf{y} = \alpha\}$. We may assume that the 5 points $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_5\}$ surround $\mathbf{x}_0$ clockwise (see *fig*.2 given below). Let $\beta = \max\{\mathbf{x}_i\cdot\mathbf{x}_j \mid 1 \le i,j \le 5, i \ne j\}$. Then $\beta \le \alpha$. We may assume $\mathbf{x}_1\cdot\mathbf{x}_2 = \beta$. Then $\beta = \frac{Q_1(j)}{3}$ and $(\mathbf{x}_1,\mathbf{x}_2) \in R_j$ with some j. Thus $p^1_{1,j} \ge 1$ holds. Since $(\mathbf{x}_0,\mathbf{x}_i) \in R_1$, the graph defined on the 5 point set $\Gamma_\alpha(\mathbf{x}_0)$ with respect to the relation R_j is regular with the degree $p^1_{1,j}$. Since β is the maximum value of the inner products between the 5 points in $\Gamma_\alpha(\mathbf{x}_0)$, $\Gamma_\alpha(\mathbf{x}_0)$ is a regular pentagon on the circle S with

$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_3 = \mathbf{x}_3 \cdot \mathbf{x}_4 = \mathbf{x}_4 \cdot \mathbf{x}_5 = \mathbf{x}_5 \cdot \mathbf{x}_1 = \beta$.
(1) Suppose that $\alpha = \beta$. $\Gamma_\alpha(\mathbf{x}_1)$ is a regular pentagon (see *fig*.2). Since $\mathbf{x}_0,\ \mathbf{x}_2,\ \mathbf{x}_5 \in \Gamma_\alpha(\mathbf{x}_1)$ and $\mathbf{x}_3,\ \mathbf{x}_4 \notin \Gamma_\alpha(\mathbf{x}_1)$, there exists $\mathbf{x}_6,\ \mathbf{x}_7 \in \Gamma_\alpha(\mathbf{x}_1)$. Since $\Gamma_\alpha(\mathbf{x}_1)$ is a regular 5-gon and α is maximum in $A(X)$, we have $\Gamma_\alpha(\mathbf{x}_1) = \{\mathbf{x}_0,\ \mathbf{x}_2,\ \mathbf{x}_6,\ \mathbf{x}_7,\ \mathbf{x}_5\}$ (which surround $\mathbf{x}_1$ counter clockwise) with $\mathbf{x}_0 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_6 = \mathbf{x}_6 \cdot \mathbf{x}_7 = \mathbf{x}_7 \cdot \mathbf{x}_5 = \mathbf{x}_5 \cdot \mathbf{x}_0 = \alpha = \beta$. Similarly $\mathbf{x}_3,\ \mathbf{x}_0,\ \mathbf{x}_1,\ \mathbf{x}_6 \in \Gamma_\alpha(\mathbf{x}_2)$ and $\mathbf{x}_4,\ \mathbf{x}_5,\ \mathbf{x}_7 \notin \Gamma_\alpha(\mathbf{x}_2)$, there exists $\mathbf{x}_8 \in \Gamma_\alpha(\mathbf{x}_2)$. Hence $\Gamma_\alpha(\mathbf{x}_2) = \{\mathbf{x}_6,\ \mathbf{x}_1,\ \mathbf{x}_0,\ \mathbf{x}_3,\ \mathbf{x}_8\}$ (which surround $\mathbf{x}_2$ counter clockwise) with $\mathbf{x}_6 \cdot \mathbf{x}_1 = \mathbf{x}_1 \cdot \mathbf{x}_0 = \mathbf{x}_0 \cdot \mathbf{x}_3 = \mathbf{x}_3 \cdot \mathbf{x}_8 = \mathbf{x}_8 \cdot \mathbf{x}_6 = \alpha$. Since the three pentagons $\Gamma_\alpha(\mathbf{x}_0)$, $\Gamma_\alpha(\mathbf{x}_1)$ and $\Gamma_\alpha(\mathbf{x}_2)$ are congruent to each other, we have $\mathbf{x}_3 \cdot \mathbf{x}_1 = \mathbf{x}_3 \cdot \mathbf{x}_5 = \mathbf{x}_3 \cdot \mathbf{x}_6$. This implies that the three points $\mathbf{x}_1, \mathbf{x}_5, \mathbf{x}_6$ are at the same distance from $\mathbf{x}_3$. We also have $\mathbf{x}_1, \mathbf{x}_5, \mathbf{x}_6 \in \Gamma_\alpha(\mathbf{x}_7)$. This implies that the two points $\mathbf{x}_3$ and $\mathbf{x}_7$ are antipodal to each other. This implies that X must be the set of vertices of a regular icosahedron with 12 vertices. Since this association scheme is antipodal, $\mathfrak{X}$ cannot be primitive, a contradiction.
(2) Suppose that $\alpha > \beta$. Since $\Gamma_\alpha(\mathbf{x}_0)$ and $\Gamma_\alpha(\mathbf{x}_1)$ are regular 5-gons on the circles of the same radii, they are congruent to each other (see *fig*.3 given below). Since $\alpha > \beta$, we have $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5 \notin \Gamma_\alpha(\mathbf{x}_1)$. Therefore $\Gamma_\alpha(\mathbf{x}_1) = \{\mathbf{x}_0,\ \mathbf{x}_6,\ \mathbf{x}_7,\ \mathbf{x}_8,\ \mathbf{x}_9\}$ (which surround $\mathbf{x}_1$ counter clockwise) with $\mathbf{x}_0 \cdot \mathbf{x}_6 = \mathbf{x}_6 \cdot \mathbf{x}_7 = \mathbf{x}_7 \cdot \mathbf{x}_8 = \mathbf{x}_8 \cdot \mathbf{x}_9 = \mathbf{x}_9 \cdot \mathbf{x}_0 = \beta$. Then $\{\mathbf{x}_0,\ \mathbf{x}_2,\ \mathbf{x}_6,\ \mathbf{x}_1\}$ are on the same plane in $\mathbb{R}^3$ and form a quadratilateral. Also the angles between the edges of the $\{\mathbf{x}_0,\ \mathbf{x}_2,\ \mathbf{x}_6,\ \mathbf{x}_1\}$ satisfy $\angle \mathbf{x}_1\mathbf{x}_0\mathbf{x}_2 = \angle \mathbf{x}_0\mathbf{x}_1\mathbf{x}_6 < \frac{2\pi}{5} < \frac{\pi}{2}$ because $\mathbf{x}_0$ and $\mathbf{x}_1$ are not on the plane determined by $\Gamma_\alpha(\mathbf{x}_0)$ and $\Gamma_\alpha(\mathbf{x}_1)$ respectively. Therefore $\mathbf{x}_2 \cdot \mathbf{x}_6 > \mathbf{x}_0 \cdot \mathbf{x}_1 = \alpha$. This contradicts the fact that α is the maximum in $A(X)$.

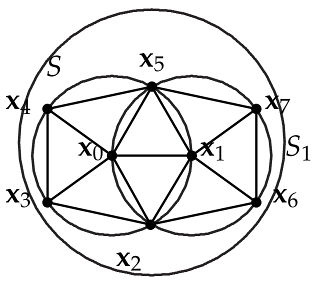

FIGURE 2. $k_1 = 5,\ \alpha = \beta$

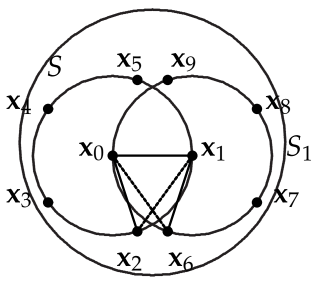

FIGURE 3. $k_1 = 5,\ \alpha > \beta$

5. THE IMPOSSIBILITY OF THE CASE $k_1 = 4$.

Let us assume that $k_1 = 4$. Fix $\mathbf{x}_0 \in X(\subset S^2 \subset \mathbb{R}^3)$. Then $\Gamma_\alpha(\mathbf{x}_0) = \{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4\}$ is on the circle $S = \{\ \mathbf{y} \in S^2 | \mathbf{x}_0 \cdot \mathbf{y} = \alpha\}$. Since nontrivial regular graphs with 4 vertices are either 4-gon or the union of 2 disjoint edges, $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4\}$ is a rectangle (which surrounds $\mathbf{x}_0$ clockwise (see *fig*.

4 below) with

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_3 \cdot \mathbf{x}_4 = \beta,$$

and

$$\mathbf{x}_2 \cdot \mathbf{x}_3 = \mathbf{x}_1 \cdot \mathbf{x}_4 = \gamma$$

Without loss of generality we may assume $\alpha \geq \beta \geq \gamma$. Let $\delta = \mathbf{x}_1 \cdot \mathbf{x}_3$ ($= \mathbf{x}_2 \cdot \mathbf{x}_4$). Then we have $\delta < \gamma$. Since $\beta = \frac{Q_1(j)}{3}$, $\gamma = \frac{Q_1(i)}{3}$ and $\delta = \frac{Q_1(l)}{3}$ with some j, i, l in $\{1, \ldots, d\}$, we have $p^1_{j,1} = p^1_{1,j} \geq 1$, $p^1_{i,1} = p^1_{1,i} \geq 1$ and $p^1_{l,1} = p^1_{1,l} \geq 1$. Then for any $\mathbf{x} \in X$, $\Gamma_\alpha(\mathbf{x})$ is also a rectangle. Let $\mathbf{u} \in \Gamma_\alpha(\mathbf{x})$. Then there must exist points $\mathbf{y}, \mathbf{z}, \mathbf{w}, \in \Gamma_\alpha(\mathbf{x})$ satisfying $\mathbf{y} \cdot \mathbf{u} = \beta$, $\mathbf{z} \cdot \mathbf{u} = \gamma$ and $\mathbf{w} \cdot \mathbf{u} = \delta$. Therefore $\Gamma_\alpha(\mathbf{x})$ must be congruent to $\Gamma_\alpha(\mathbf{x}_0)$ for any $\mathbf{x} \in X$.

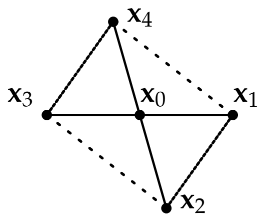

FIGURE 4.
$\alpha \geq \beta \geq \gamma$

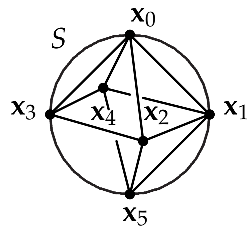

FIGURE 5.
$\alpha = \beta = \gamma$

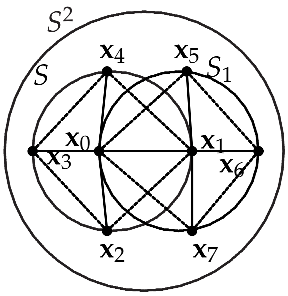

FIGURE 6.
$\alpha > \beta = \gamma$

(1) Suppose that $\alpha = \beta = \gamma$. Then $\Gamma_\alpha(x_0) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is a square with $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_3 = \mathbf{x}_3 \cdot \mathbf{x}_4 = \mathbf{x}_4 \cdot \mathbf{x}_1 = \alpha$. Since $\mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_4 \in \Gamma_\alpha(\mathbf{x}_1)$ and $\mathbf{x}_3 \notin \Gamma_\alpha(\mathbf{x}_1)$, there exists $\mathbf{x}_5 \in \Gamma_\alpha(\mathbf{x}_1)$. Since $\Gamma_\alpha(\mathbf{x}_1)$ is congruent to the square $\Gamma_\alpha(\mathbf{x}_0)$, the point $\mathbf{x}_5$ is at the same distance from the 3 points $\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4$ (see *fig*.5 given below). This implies that $\mathbf{x}_0$ and $\mathbf{x}_5$ form an antipodal pair on S^2. Then X must be the set of vertices of a regular octahedron of 6 vertices. Since this association scheme is antipodal, $\mathfrak{X}$ cannot be primitive, a contradiction.

(2) Suppose that $\alpha > \beta = \gamma$. Then $\Gamma_\alpha(\mathbf{x}_0) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ forms a

square with $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_3 = \mathbf{x}_3 \cdot \mathbf{x}_4 = \mathbf{x}_4 \cdot \mathbf{x}_1 = \beta$. Since $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \notin \Gamma_\alpha(\mathbf{x}_1)$ and $\mathbf{x}_0 \in \Gamma_\alpha(\mathbf{x}_1)$, there exists 3 points $\mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7$ in $\Gamma_\alpha(\mathbf{x}_1)$. Since $\Gamma_\alpha(\mathbf{x}_1)$ is congruent to $\Gamma_\alpha(\mathbf{x}_0)$, $\{\mathbf{x}_0, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7\}$ surround $\mathbf{x}_1$ clockwise with $\mathbf{x}_0 \cdot \mathbf{x}_5 = \mathbf{x}_5 \cdot \mathbf{x}_6 = \mathbf{x}_6 \cdot \mathbf{x}_7 = \mathbf{x}_7 \cdot \mathbf{x}_0 = \beta$ (see *fig*.6 above). Then the 4 points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_7$ are on the same plane of $\mathbb{R}^3$. Since $\angle \mathbf{x}_2\mathbf{x}_0\mathbf{x}_1 = \angle \mathbf{x}_7\mathbf{x}_1\mathbf{x}_0 < \frac{\pi}{2}$, we have $\mathbf{x}_2 \cdot \mathbf{x}_7 > \mathbf{x}_0 \cdot \mathbf{x}_1 = \alpha$ holds. This contradicts the fact that α is the maximum value in $A(X)$.
(3) Suppose that $\alpha = \beta > \gamma$. Then $\Gamma_\alpha(\mathbf{x}_0) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is a rectangle (which surrounds x_0 clockwise) with $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_3 \cdot \mathbf{x}_4 = \alpha$ and $\mathbf{x}_2 \cdot \mathbf{x}_3 = \mathbf{x}_4 \cdot \mathbf{x}_1 = \gamma$. Since $\mathbf{x}_0, \mathbf{x}_2 \in \Gamma_\alpha(\mathbf{x}_1)$ and $\mathbf{x}_3, \mathbf{x}_4 \notin \Gamma_\alpha(\mathbf{x}_1)$, there exist 2 points $\mathbf{x}_5, \mathbf{x}_6 \in \Gamma_\alpha(\mathbf{x}_1)$. Since $\Gamma_\alpha(\mathbf{x}_1)$ is congruent to $\Gamma_\alpha(\mathbf{x}_0)$ and $\mathbf{x}_0 \cdot \mathbf{x}_2 = \alpha, \Gamma_\alpha(\mathbf{x}_1) = \{\mathbf{x}_0, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_2\}$ (which surrounds x_1 clockwise) with $\mathbf{x}_0 \cdot \mathbf{x}_2 = \mathbf{x}_5 \cdot \mathbf{x}_6 = \alpha$ and $\mathbf{x}_2 \cdot \mathbf{x}_6 = \mathbf{x}_0 \cdot \mathbf{x}_5 = \gamma$ (see *fig*.7 below). Then we have $\mathbf{x}_0 \cdot \mathbf{x}_6 = \mathbf{x}_2 \cdot \mathbf{x}_5 = \delta$. The three rectangles $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4$, $\mathbf{x}_2\mathbf{x}_0\mathbf{x}_5\mathbf{x}_6$, $\mathbf{x}_0\mathbf{x}_1\mathbf{x}_7\mathbf{x}_8$ move to each other by rotations around the line passing through the origin of the sphere and perpendicular to the regular triangle $\mathbf{x}_1\mathbf{x}_0\mathbf{x}_2$. Therefor each of the following three 4 point sets are on the same plane in $\mathbb{R}^3$:

$$\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_5, \mathbf{x}_4\}, \quad \{\mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_3, \mathbf{x}_8\}, \quad \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_7, \mathbf{x}_6\}.$$

Also the 6 point set $\{\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7, \mathbf{x}_8\}$ is on the same plane.
(a) First we assume $\mathbf{x}_4 \cdot \mathbf{x}_5 = \alpha$. Then $\mathbf{x}_0\mathbf{x}_1\mathbf{x}_5\mathbf{x}_4$, $\mathbf{x}_2\mathbf{x}_0\mathbf{x}_3\mathbf{x}_8$, $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_7\mathbf{x}_6$ are squares and $\mathbf{x}_3\mathbf{x}_4\mathbf{x}_5\mathbf{x}_6\mathbf{x}_7\mathbf{x}_8$ is a regular hexagon (see *fig*.8 below).
Since $\mathbf{x}_8 \in \Gamma_\alpha(\mathbf{x}_3) \cap \Gamma_\alpha(\mathbf{x}_2) \cap \Gamma_\alpha(\mathbf{x}_7)$, we have $\mathbf{x}_8 \cdot \mathbf{x}_4 = \mathbf{x}_8 \cdot \mathbf{x}_1 = \mathbf{x}_8 \cdot \mathbf{x}_6 = \delta$. Thus $\mathbf{x}_8$ is at the same distance from the three points $\mathbf{x}_4, \mathbf{x}_1, \mathbf{x}_6$. On the other hand $\mathbf{x}_5$ is also at the same distance from the three points $\mathbf{x}_4, \mathbf{x}_1, \mathbf{x}_6$. Since $\mathbf{x}_5$ and $\mathbf{x}_8$ are at the opposite side of the plane containing the triangle $\mathbf{x}_4\mathbf{x}_1\mathbf{x}_6$, they are an antipodal pair of S^2. This implies that X must be the set of all the 12 vertices of a quasi-regular polyhedron of type $[3,4,3,4]$. However, this is impossible because the quasi-regular polyhedron of type $[3,4,3,4]$ is an antipodal set.
(b) Next we assume $\mathbf{x}_4 \cdot \mathbf{x}_5 = \mathbf{x}_6 \cdot \mathbf{x}_7 = \mathbf{x}_8 \cdot \mathbf{x}_3 < \alpha$. Since the six points $\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7, \mathbf{x}_8$ are on a circle, we have $\mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7 \notin \Gamma_\alpha(\mathbf{x}_3)$. Since $\mathbf{x}_1, \mathbf{x}_2 \notin \Gamma_\alpha(\mathbf{x}_3)$, there exists 2 points $\mathbf{x}_9, \mathbf{x}_{10} \in \Gamma_\alpha(\mathbf{x}_3)$ (see *fig*.9 given above). Similarly there exist 2 points $\mathbf{x}_{11}, \mathbf{x}_{12} \in \Gamma_\alpha(\mathbf{x}_4)$. Then both $\mathbf{x}_8$ and $\mathbf{x}_9$ must be on the plane determined by the three points $x_2, \mathbf{x}_0, \mathbf{x}_3$. Hence the five points $\mathbf{x}_8, \mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_3, \mathbf{x}_9$ must be on a same plane.
(b-1) Assume that $\mathbf{x}_8 \cdot \mathbf{x}_9 = \alpha$ holds. Then $\mathbf{x}_8\mathbf{x}_2\mathbf{x}_0\mathbf{x}_3\mathbf{x}_9$ is a regular pentagon. Similarly $\mathbf{x}_0\mathbf{x}_1\mathbf{x}_5\mathbf{x}_{12}\mathbf{x}_4$ is a regular pentagon which is congruent to $\mathbf{x}_8\mathbf{x}_2\mathbf{x}_0\mathbf{x}_3\mathbf{x}_9$. Thus we can show that each edge of a regular triangle is adjacent to a regular pentagon (see *fig*.9). Then the two points $\mathbf{a}$ and $\mathbf{b}$ in *fig*.9 are antipodal to each other. Therefore X is the quasi-regular polyhedron $[3,5,3,5]$ of 30 vertices (icosidodecahedron). However, since the quasi-regular polyhedron $[3,5,3,5]$ is antipodal, this is impossible.
(b-2) Next, assume that $\mathbf{x}_8 \cdot \mathbf{x}_9 < \alpha$ holds. Then there must exist $\mathbf{u}, \mathbf{v} \in \Gamma_\alpha(\mathbf{x}_8)$ satisfying $\mathbf{x}_9 \neq \mathbf{u}, \mathbf{v}$ (see *fig*.10. below). Then $\mathbf{u}$ must be on the same

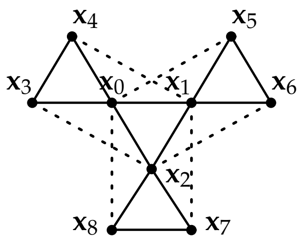

FIGURE 7.
$\alpha = \beta > \gamma$

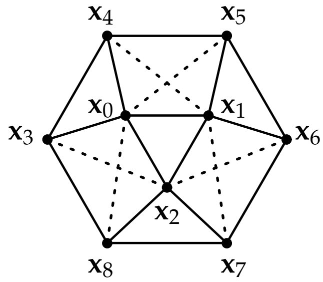

FIGURE 8.
(a) $\mathbf{x}_4 \cdot \mathbf{x}_5 = \alpha$

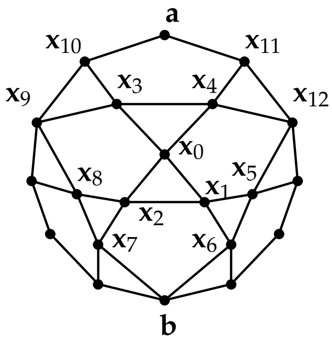

FIGURE 9.
(b-1) $\mathbf{x}_4 \cdot \mathbf{x}_5 < \alpha$, $\mathbf{x}_8 \cdot \mathbf{x}_9 = \alpha$

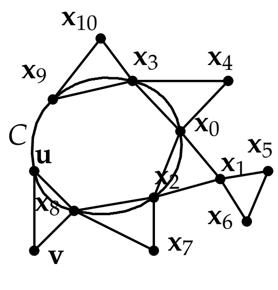

FIGURE 10

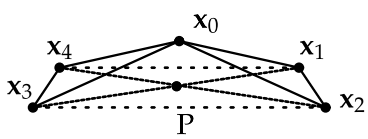

FIGURE 11

plane determined by $\{\mathbf{x}_8, \mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_3, \mathbf{x}_9\}$. Thus $\mathbf{u}, \mathbf{x}_8, \mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_3, \mathbf{x}_9$ is on a circle C in S^2.

Let us consider the rectangular cone $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ (see *fig*.11 given above). Let P be the center of the rectangle $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4$. Then we have

$$\angle \mathbf{x}_1 \mathrm{P} \mathbf{x}_2 > \angle \mathbf{x}_1 \mathbf{x}_0 \mathbf{x}_2 = \frac{\pi}{3}.$$

Hence we have

$$\angle \mathbf{x}_2 \mathbf{x}_0 \mathbf{x}_3 < \angle \mathbf{x}_3 \mathrm{P} \mathbf{x}_2 < \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Similarly we have

$$\angle \mathbf{u}\mathbf{x}_8\mathbf{x}_2 = \angle \mathbf{x}_8\mathbf{x}_2\mathbf{x}_0 = \angle \mathbf{x}_0\mathbf{x}_3\mathbf{x}_9 < \frac{2\pi}{3}.$$

Since $\mathbf{u} \cdot \mathbf{x}_9 \leq \alpha$, this implies that the length of every edge of the hexagon $\mathbf{u}\mathbf{x}_8\mathbf{x}_2\mathbf{x}_0\mathbf{x}_3\mathbf{x}_9$ is longer than the radius of the circle C. This is a contradiction. Therefore this case does not occur.

(4) Suppose $\alpha > \beta > \gamma$.

Let us consider the rectangular cone $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$. Let P be the center of the base lectangle (see *fig*.12 given below). Then $\angle \mathbf{x}_1\mathbf{x}_0\mathbf{x}_2 < \angle \mathbf{x}_1 \mathrm{P} \mathbf{x}_2 < \frac{\pi}{2}$. Since $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \notin \Gamma_\alpha(\mathbf{x}_0)$, there exist $\mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7 \in \Gamma_\alpha(\mathbf{x}_1)$ (see *fig*.13 given below). If $\mathbf{x}_0 \cdot \mathbf{x}_7 = \beta$, then $\mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_7$ must be on the same plane. Moreover $\angle \mathbf{x}_0\mathbf{x}_1\mathbf{x}_7 = \angle \mathbf{x}_1\mathbf{x}_0\mathbf{x}_2 < \frac{\pi}{2}$ holds. Hence $\mathbf{x}_2 \cdot \mathbf{x}_7 > \alpha$ holds. But this is impossible. Therefore we have $\mathbf{x}_0 \cdot \mathbf{x}_7 = \gamma$. Similar consideration for $\Gamma(\mathbf{x}_2), \Gamma(\mathbf{x}_3), \Gamma(\mathbf{x}_4)$ will yield three more rectangles (see

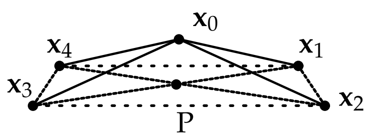

FIGURE 12.
$\alpha > \beta > \gamma$

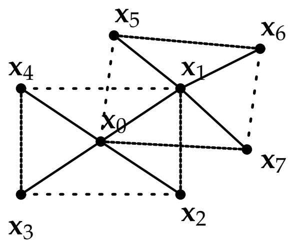

FIGURE 13.
$\mathbf{x}_0 \cdot \mathbf{x}_7 = \beta$

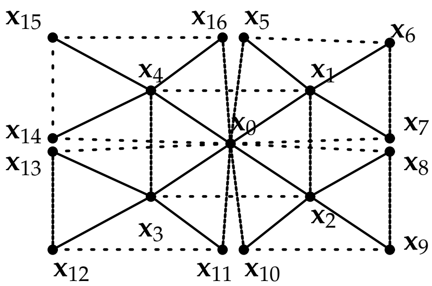

FIGURE 14.
$\mathbf{x}_0 \cdot \mathbf{x}_7 = \gamma$

fig.14 given above) having $\mathbf{x}_0$ in common. Where we may possibly have $\mathbf{x}_7 = \mathbf{x}_8$, $\mathbf{x}_{10} = \mathbf{x}_{11}$, $\mathbf{x}_{13} = \mathbf{x}_{14}$, $\mathbf{x}_{16} = \mathbf{x}_5$. However, this is a contradiction because $\angle \mathbf{x}_5\mathbf{x}_0\mathbf{x}_7 = \angle \mathbf{x}_8\mathbf{x}_0\mathbf{x}_{10} = \angle \mathbf{x}_{11}\mathbf{x}_0\mathbf{x}_{13} = \angle \mathbf{x}_{14}\mathbf{x}_0\mathbf{x}_{16} = \frac{\pi}{2}$.

Thus we have shown that $k_1 = 4$ is immpossible.

6. THE CASE $k_1 = 3$.

Let us assume $k_1 = 3$. Fix $\mathbf{x}_0 \in X \subset S^2$. Then $\Gamma_\alpha(\mathbf{x}_0) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a regular triangle which surrounds $\mathbf{x}_0$ clockwise (see *fig*.15 given below). Then we must have $\angle \mathbf{x}_1\mathbf{x}_0\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}_0\mathbf{x}_3 = \mathbf{x}_3\mathbf{x}_0\mathbf{x}_1 < \frac{2\pi}{3}$.
(1) If $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_3 = \mathbf{x}_3 \cdot \mathbf{x}_1 = \alpha$, then $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ forms a regular tetrahedron.
(2) If $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_3 = \mathbf{x}_3 \cdot \mathbf{x}_1 < \alpha$, then $\mathbf{x}_2, \mathbf{x}_3 \notin \Gamma_\alpha(\mathbf{x}_1)$. Hence there exist $\mathbf{x}_4, \mathbf{x}_5 \in \Gamma_\alpha(\mathbf{x}_1)$ (see *fig*.15 given below). Then the 4 points $\{\mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_5\}$ must be on the same plane. Therefore $\{\mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_5\}$ is on a circle C on the sphere S^2.
(a) If $\mathbf{x}_2 \cdot \mathbf{x}_5 = \alpha$, then the four isosceles triangles $\mathbf{x}_1\mathbf{x}_0\mathbf{x}_5, \mathbf{x}_0\mathbf{x}_2\mathbf{x}_1, \mathbf{x}_2\mathbf{x}_5\mathbf{x}_0$, and $\mathbf{x}_5\mathbf{x}_4\mathbf{x}_1$ are isometric to each other. Hence $\{\mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_5\}$ must form a square. Then we must have $\mathbf{x}_3 \cdot \mathbf{x}_4 = \alpha$. We can easily show that X is a set of all the 8 vertices of a cube. However, since cubes are antipodal, this is impossible.
(b) Next we assume $\mathbf{x}_2 \cdot \mathbf{x}_5 < \alpha$. Then there exist $\mathbf{x}_6, \mathbf{x}_7 \in \Gamma_\alpha(\mathbf{x}_2)$ (see *fig*.17 given below). Since $\mathbf{x}_6, \mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_1$ are on the same plane, $\mathbf{x}_6$ must be on the circle C.
(b-1) Assume $\mathbf{x}_6 \cdot \mathbf{x}_5 = \alpha$. Then $\mathbf{x}_6\mathbf{x}_2\mathbf{x}_0\mathbf{x}_1\mathbf{x}_5$ is a regular pentagon.

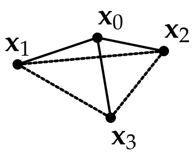

FIGURE 15

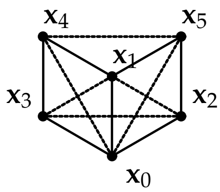

FIGURE 16. $\mathbf{x}_1 \cdot \mathbf{x}_2 < \alpha$ $\mathbf{x}_2 \cdot \mathbf{x}_5 = \alpha$

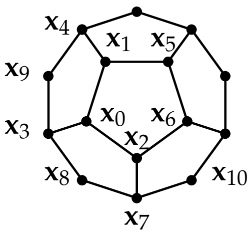

FIGURE 17. $\mathbf{x}_2 \cdot \mathbf{x}_5 < \alpha$ $\mathbf{x}_5 \cdot \mathbf{x}_6 = \alpha$

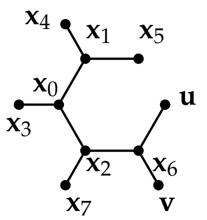

FIGURE 18. $\mathbf{x}_2 \cdot \mathbf{x}_5 < \alpha$ $\mathbf{x}_5 \cdot \mathbf{x}_6 < \alpha$

Then there are 5 regular pentagons isometric to $\mathbf{x}_6\mathbf{x}_2\mathbf{x}_0\mathbf{x}_1\mathbf{x}_5$, attached to $\mathbf{x}_6\mathbf{x}_2\mathbf{x}_0\mathbf{x}_1\mathbf{x}_5$ (see *fig*.17 given above). Then $\mathbf{x}_4$ and $\mathbf{x}_{10}$ are antipodal to each other (see *fig*.17 given above). Therefore X is the set of all the 20 vertices of the regular dodecahedron. However, since a dodecahedron is antipodal, this is impossible.
(b-2) Finally, assume $\mathbf{x}_5 \cdot \mathbf{x}_6 < \alpha$. Then there exist $\mathbf{u}, \mathbf{v} \in \Gamma_\alpha(\mathbf{x}_6)$ (see *fig*.18 given above). Since $\mathbf{x}_0, \mathbf{x}_2, \mathbf{x}_6, \mathbf{u}$ are on a same plane $\mathbf{u}$ is on the circle C. Since $\angle \mathbf{x}_5\mathbf{x}_1\mathbf{x}_0 = \mathbf{x}_0\mathbf{x}_2\mathbf{x}_6 = \mathbf{x}_2\mathbf{x}_6\mathbf{u} = \mathbf{x}_1\mathbf{x}_0\mathbf{x}_2 < \frac{2\pi}{3}$, the length of the edges $\mathbf{x}_5\mathbf{x}_1$, $\mathbf{x}_1\mathbf{x}_0$, $\mathbf{x}_0\mathbf{x}_2$, $\mathbf{x}_2\mathbf{x}_6$, $\mathbf{x}_6\mathbf{u}$ are equal and longer than the length of the edge of the regular hexagon on C. Therefore the length of the edge $\mathbf{x}_5\mathbf{u}$ is less than that of $\mathbf{x}_5\mathbf{x}_1$. This implies $\mathbf{x}_5 \cdot \mathbf{u} > \alpha$. This is a contradiction.

We remark that this case $k_1 = 3$ was originally treated by using the result of Yamazaki [8] which classifies symmetric association schemes with

$k_1 = 3$. Our present treatment avoids the use of this difficult and deep result of Yamazaki.

7. Completion of the proof of Theorem 1.

We have shown that $k_1 = 5, 4$ are impossible in Sections 4 and 5. The case $k \leq 2$ is also impossible (see Proposition 4 in Section 2). Also we have shown in Section 6 that if $k_1 = 3$, the only possibility for X is the set of all the 4 vertices of a tetrahedron. Hence we have Theorem 1.

Remarks. It would be interesting to weaken some of the assumptions of Theorem 1 and then to classify such association schemes. For example, it would be interesting to classify imprimitive symmetric association schemes with $m_1 = 3$. Also, it would be interesting to classify primitive (non symmetric) commutative association schemes with $m_1 = 3$. This has been already treated in Hirasaka [5], while our present paper was being revised. Of course, it would be interesting if one could classify primitive symmetric association schemes for other small values of m_1, say 4. It seems that it is possible to classify symmetric Q-polynomial association schemes with $m_1 = 4$ by generalizing the ideas employed in the present paper. We hope that we can come back to this problem in the near future.

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