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BEST SIMULTANEOUS DIOPHANTINE APPROXIMATIONS UNDER A CONSTRAINT ON THE DENOMINATOR

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ABSTRACT. We investigate the problem of best simultaneous Diophantine approximation under a constraint on the denominator, as proposed by Jurkat. New lower estimates for optimal approximation constants are given in terms of critical determinants of suitable star bodies. Tools are results on simultaneous Diophantine approximation of rationals by rationals with smaller denominator. Finally, the approximation results are applied to the decomposition of integer vectors.

1. INTRODUCTION

The first investigations of simultaneous Diophantine approximation with constraints on the denominator are due to Jurkat [10]. Kratz [11, 12] considered the following particular problem: let $x \in \mathbb{R}^k$, $k \geq 2$, and $g(\cdot) = ||\cdot||_2$. As in Kratz [11], define for $Q > 0$ the *successive minima* $\lambda_i = \lambda_i(x, Q)$, $i = 1, \ldots, k+1$, *of* x *under the constraint* $|q| \leq Q$ as follows: λ_i is the minimum of all $\lambda \geq 0$, for which there are i linearly independent vectors $p_j = (p_{j1}, \ldots, p_{jk}, p_{jk+1}) \in \mathbb{Z}^{k+1}$, $j = 1, \ldots, i$, such that

$$g(p_{jk+1}x - (p_{j1}, \ldots, p_{jk})) \leq \lambda \text{ and } |p_{jk+1}| \leq Q \text{ for } j = 1, \ldots, i.$$

It is known (see e. g. [11]) that the product of the first k successive minima satisfies

$$\lambda_1 \cdots \lambda_k = O\left(\frac{1}{Q}\right).$$

In this paper we are interested in an optimal constant $c = c(k, ||\cdot||_2)$ such that

$$\lambda_1 \cdots \lambda_k < \frac{c}{Q}.$$

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Kratz proved in [12] that

$$c(2, ||\cdot||_2) = \frac{2}{\sqrt{3}}.$$

Assume now, that $g(\cdot)$ is the distance function of a bounded star body K in $\mathbb{R}^k$. In the following we consider the above problem for $g(\cdot)$ and show in Theorem 1.5 that

$$c(k, g) \geq \frac{1}{\Delta(K)},$$

where $\Delta(K)$ is the critical determinant of K. Let γ_k denote the Hermite constant. Since the critical determinant of the k-dimensional unit ball equals $\gamma_k^{-k/2}$, we conclude that

$$c(k, ||\cdot||_2) \geq \gamma_k^{\frac{k}{2}}.$$

For recent results on the upper bound for $c(k, g)$, see [3].

To obtain these results, we study in detail the simultaneous approximation of rational numbers by rational numbers with smaller denominator. Let $n = (n_1, \ldots, n_k, n_{k+1}) \in \mathbb{Z}^{k+1}$, $k \geq 2$, be an integer vector. Assume that $0 < n_1 \leq \ldots \leq n_{k+1}$ and that $\gcd(n_1, \ldots, n_{k+1}) = 1$. Consider the problem of approximating the rational vector $(n_1/n_{k+1}, \ldots, n_k/n_{k+1})$ by rational vectors of the form $(m_1/m_{k+1}, \ldots, m_k/m_{k+1})$ with $m_i \in \mathbb{Z}$, $i = 1, \ldots, k+1$, and $0 \leq m_{k+1} < n_{k+1}$. More precisely, we investigate the behavior of the points

$$\left(m_1 - m_{k+1}\frac{n_1}{n_{k+1}}, \ldots, m_k - m_{k+1}\frac{n_k}{n_{k+1}}\right) \tag{1}$$

as $m = (m_1, \ldots, m_k, m_{k+1})$ ranges over $\mathbb{Z}^{k+1}$. Since these points form a k–dimensional lattice $\Lambda(n)$ (see Section 2 for details), we make use of tools from the geometry of numbers.

Given an arbitrary lattice $\Lambda \subset \mathbb{Q}^k$, we can construct a sequence of integer vectors $n(t)$ such that the sequence of corresponding lattices $\Lambda(n(t))$ after an appropriate normalization tends to Λ.

Theorem 1.1. *For any rational lattice Λ with basis $\{b_1, \ldots, b_k\}$, $b_i \in \mathbb{Q}^k$, $i = 1, \ldots, k$ and for all rationals $\alpha_1, \ldots, \alpha_k$ with $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq 1$, there exists an arithmetic sequence $\mathcal{P}$ and a sequence*

$$n(t) = (n_1(t), \ldots, n_k(t), n_{k+1}(t)) \in \mathbb{Z}^{k+1}, t \in \mathcal{P},$$

such that

$$\gcd(n_1(t), \ldots, n_k(t), n_{k+1}(t)) = 1$$

and $\Lambda(n(t))$ has a basis $a_1(t), \ldots, a_k(t)$ with

$$a_{ij}(t) = \frac{b_{ij}}{d\,t} + O\left(\frac{1}{t^2}\right) \text{ for } i, j = 1, \ldots, k, \tag{2}$$

where $d \in \mathbb{N}$ is such that $d\, b_{ij}, d\, \alpha_j\, b_{ij} \in \mathbb{Z}$ for all $i, j = 1, \ldots, k$. Moreover,

$$n_{k+1}(t) = \frac{d^k t^k}{\det \Lambda} + O(t^{k-1}) \tag{3}$$

and

$$\alpha_i(t) := \frac{n_i(t)}{n_{k+1}(t)} = \alpha_i + O\left(\frac{1}{t}\right). \tag{4}$$

Let $\alpha(K)$ denote the anomaly of a set K, and if f is the distance function of K, then both $\lambda_i(f, \Lambda)$ and $\lambda_i(K, \Lambda)$ denote the i th successive minimum of the lattice Λ with respect to the set K.

Theorem 1.2. *Let K be a bounded star body in $\mathbb{R}^k$ and let*

$$\mathbb{U}^{k+1} = \{x \in \mathbb{Z}^{k+1} : 0 < x_1 \leq \cdots \leq x_{k+1},\ \gcd(x_1, \ldots, x_{k+1}) = 1\}.$$

Then

$$C(K) := \sup_{n \in \mathbb{U}^{k+1}} \frac{\lambda_1(K, \Lambda(n)) \cdots \lambda_k(K, \Lambda(n))}{\det \Lambda(n)} = \frac{\alpha(K)}{\Delta(K)}.$$

Moreover, for all $\alpha_1, \ldots, \alpha_k \in \mathbb{Q}$ with $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq 1$, there exists an infinite sequence of integer vectors $n(t) = (n_1(t), \ldots, n_k(t), n_{k+1}(t)) \in \mathbb{U}^{k+1}$, $t \in T = \{t_1, t_2, \ldots\}$, such that

$$\lim_{\substack{t \to \infty \\ t \in T}} \frac{\lambda_1(K, \Lambda(n(t))) \cdots \lambda_k(K, \Lambda(n(t)))}{\det \Lambda(n(t))} = C(K); \tag{5}$$

$$\lim_{\substack{t \to \infty \\ t \in T}} \frac{n_i(t)}{n_{k+1}(t)} = \alpha_i,\ i = 1, \ldots, k. \tag{6}$$

and

$$\lim_{\substack{t \to \infty \\ t \in T}} n_{k+1}(t) = \infty. \tag{7}$$

The proof of Theorem 1.2 is based on the following lemma which is of independent interest. Let D denote the set of functions $f : \mathbb{R}^k \to [0, +\infty)$ which are positive homogeneous of degree 1. We also denote by o the zero vector.

Lemma 1.3. *Let $\{f_t\}$ be a sequence of functions in D which converges uniformly on $||x|| \leq 1$ to a function f in D and let $\{L_t\}$ be a sequence of lattices in $\mathbb{R}^k$ which converges to a lattice L. Then*

(i) $\limsup_{t \to \infty} \lambda_i(f_t, L_t) \leq \lambda_i(f, L)$, *for $i = 1, \ldots, k$;*

(ii) *If, in addition, $f(x) = 0$ only for $x = o$, then $\lim_{t \to \infty} \lambda_i(f_t, L_t)$ exists and equals $\lambda_i(f, L)$ for $i = 1, \ldots, k$.*

This lemma clearly implies the following result.

Corollary 1.4. *If $\{L_t\}$ is a sequence of lattices in $\mathbb{R}^k$ convergent to a full lattice L and K is a bounded star body then*

$$\lim_{t\to\infty} \lambda_i(K, L_t) = \lambda_i(K, L) \text{ for each } i = 1, \ldots, k.$$

A similar result about centrally symmetric convex bodies was recently proved by the first author jointly with Schinzel and Schmidt in [4].

Theorem 1.5. *Let $g(\cdot)$ be the distance function of a bounded star body K in $\mathbb{R}^k$. Then*

$$c(k, g) \geq \frac{1}{\Delta(K)}.$$

Theorem 1.1 will be applied in Section 8 to the problem of decomposition of integer vectors, where the problem is considered with respect to the supremum norm. For recent results on this problem for the Euclidean norm $||\cdot||_2$, see [4]. By tradition, we denote the supremum norm of a vector a by $h(a)$.

Given m linearly independent vectors $n_1, \ldots, n_m$ in $\mathbb{Z}^{k+1}$ let $H(n_1, \ldots, n_m)$ denote the maximum of the absolute values of the $m \times m$–minors of the matrix $(n_1^t, \ldots, n_m^t)$ and let $D(n_1, \ldots, n_m)$ be the greatest common divisor of these minors. Then $h(n) = H(n)$ for $n \neq o$. For $k+1 > l > m > 0$ let

$$c_0(k+1, l, m) = \sup \inf \left(\frac{D(n_1, \ldots, n_m)}{H(n_1, \ldots, n_m)} \right)^{\frac{k-l+1}{k-m+1}} \prod_{i=1}^{l} h(p_i), \tag{8}$$

where the supremum is taken over all sets of linearly independent vectors $n_1, \ldots, n_m$ in $\mathbb{Z}^{k+1}$ and the infimum over all sets of linearly independent vectors $p_1, \ldots, p_l$ in $\mathbb{Z}^{k+1}$ such that

$$n_i = \sum_{j=1}^{l} u_{ij} p_j, u_{ij} \in \mathbb{Q} \text{ for all } i \leq m.$$

It has been proved in [14] that for fixed l, m,

$$\limsup_{k\to\infty} c_0(k+1, l, m) < \infty \tag{9}$$

and in [2] it was shown that

$$c_0(k+1, 2, 1) \leq \frac{2}{(k+1)^{\frac{1}{k}}}.$$

A result in [7] says that $c_0(3, 2, 1) = 2/\sqrt{3}$. Note that

$$c_0(k+1, 2, 1) = \sup_{n \in \mathbb{Z}^{k+1} \setminus \{o\}} \inf_{\substack{p,q \in \mathbb{Z}^{k+1} \setminus \{o\} \\ \dim(p,q)=2 \\ n = up+vq, u,v \in \mathbb{Z}}} \frac{h(p)h(q)}{h(n)^{1-\frac{1}{k}}}.$$

In this paper we continue to study the behavior of $c_0(k+1,2,1)$ and prove the following theorem.

Theorem 1.6. *For $k \geq 3$*

$$\limsup_{\substack{n\in\mathbb{Z}^{k+1}\\ h(n)\to\infty}} \inf_{\substack{p,q\in\mathbb{Z}^{k+1}\\ \dim(p,q)=2\\ n=up+vq,u,v\in\mathbb{Z}}} \frac{h(p)h(q)}{h(n)^{1-\frac{1}{k}}} \geq \frac{1}{(k+1)^{\frac{1}{k}}}.$$

A more general result of Chaładus [6] yields a weaker inequality with $1/2$ instead of $1/(k+1)^{1/k}$.

2. The Lattice $\Lambda(n)$, Rational Weyl Sequences and Systems of Linear Congruences

In this section we construct a special lattice $\Lambda(n)$. Its points correspond to points of the form (1). Given the vector n, there is a basis of the lattice $\mathbb{Z}^{k+1}$ of the form $n, v_1, \ldots, v_k$. Let

$$v_i' = \left(v_{i1} - v_{ik+1}\frac{n_1}{n_{k+1}}, \ldots, v_{ik} - v_{ik+1}\frac{n_k}{n_{k+1}}\right) \in \mathbb{R}^k, i = 1, \ldots, k.$$

The equality

$$A_1 v_1' + \ldots + A_k v_k' = o$$

implies that

$$n_{k+1}A_1 v_1 + \ldots + n_{k+1}A_k v_k + A_{k+1} n = o$$

with $A_{k+1} = -A_1 v_{1k+1} - \ldots - A_k v_{kk+1}$. Thus the vectors $v_1', \ldots, v_k'$ are linearly independent. Denote by $\Lambda(n)$ the k–dimensional lattice with basis $\{v_1', \ldots, v_k'\}$. Since

$$1 = \det\begin{pmatrix} v_{11} & \ldots & v_{k1} & n_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{1k} & \ldots & v_{kk} & n_k \\ v_{1k+1} & \ldots & v_{kk+1} & n_{k+1} \end{pmatrix}$$

$$= n_{k+1}\det\begin{pmatrix} v_{11} - v_{1k+1}\frac{n_1}{n_{k+1}} & \ldots & v_{k1} - v_{kk+1}\frac{n_1}{n_{k+1}} & \frac{n_1}{n_{k+1}} \\ \vdots & \ddots & \vdots & \vdots \\ v_{1k} - v_{1k+1}\frac{n_k}{n_{k+1}} & \ldots & v_{kk} - v_{kk+1}\frac{n_k}{n_{k+1}} & \frac{n_k}{n_{k+1}} \\ 0 & \ldots & 0 & 1 \end{pmatrix},$$

we have $\det\Lambda(n) = 1/n_{k+1}$. It is easily seen, that for every non–zero vector $v \in \Lambda(n)$ there is a unique vector $m \in \mathbb{Z}^{k+1}$ such that

$$v = \left(m_1 - m_{k+1}\frac{n_1}{n_{k+1}}, \ldots, m_k - m_{k+1}\frac{n_k}{n_{k+1}}\right), \text{ where } 0 \leq m_{k+1} < n_{k+1}.$$

Thus there is a one–to–one correspondence between the points of $\Lambda(n) \setminus \{o\}$ and the non–zero integer vectors with $0 \le m_{k+1} < n_{k+1}$. Note also that since $v \neq o$, the vectors m and n are linearly independent.

If Λ is a lattice, let Λ^* be its *polar lattice*, see [8]. The lattice $\Lambda(n)$ is related to the lattice $\Lambda^\perp(n)$ of integer vectors orthogonal to n. Let $\Lambda^\perp_{k+1}(n)$ be the k-dimensional lattice obtained by omitting the $(k+1)$st coordinate in $\Lambda^\perp(n)$. Then the following holds:

Lemma 2.1. *The lattice $\Lambda^\perp_{k+1}(n)$ is the polar lattice of the lattice $\Lambda(n)$,*

$$\Lambda^\perp_{k+1}(n) = \Lambda(n)^*.$$

The lattice $\Lambda(n)$ appears in some problems of number theory. Let $\theta_1, \dots, \theta_k$, $k \ge 2$, be real numbers and let $\mathcal{W}_k$ be the sequence of k–dimensional vectors

$$(i\theta_1 \mod 1, \dots, i\theta_k \mod 1),\ i = 0, 1, 2 \dots \tag{10}$$

$\mathcal{W}_k$ is called a k–dimensional *Weyl sequence*. We shall consider the case where

$$\theta_1 = \frac{n_1}{n_{k+1}}, \dots, \theta_k = \frac{n_k}{n_{k+1}}.$$

Then $\mathcal{W}_k$ is n_{k+1}–periodic and the set

$$\Lambda(\mathcal{W}_k) = \{x + y : x \in \mathbb{Z}^k,\ y \in \mathcal{W}_k\}$$

is a k–dimensional lattice. It can be shown easily that

$$\Lambda(\mathcal{W}_k) = \Lambda(n).$$

Consider the lattice $n_{k+1}\Lambda(n) = n_{k+1}\Lambda(\mathcal{W}_k) \subset \mathbb{Z}^k$. The points in (10), multiplied by n_{k+1}, can be written in the form

$$(in_1 \mod n_{k+1}, \dots, in_k \mod n_{k+1}),\ i = 0, 1, 2, \dots$$

Therefore, any point $(x_1, \dots, x_k) \in n_{k+1}\Lambda(n)$ is a solution of the system

$$\begin{cases} x_1 + rn_1 & \equiv & 0 \pmod{n_{k+1}} \\ \quad\vdots & & \\ x_k + rn_k & \equiv & 0 \pmod{n_{k+1}} \end{cases} \tag{11}$$

where r is an integer corresponding to m_{k+1}. Hence we may consider Theorems 1.1 and 1.2 as results on rational Weyl sequences and solutions of the system (11).

3. Proof of Lemma 2.1

Let v be a primitive non–zero vector of $\Lambda(n)$ and $V = n_{k+1}v$. Choose a vector $m \in \mathbb{Z}^{k+1}$ such that

$$v = \left(m_1 - m_{k+1}\frac{n_1}{n_{k+1}}, \dots, m_k - m_{k+1}\frac{n_k}{n_{k+1}}\right).$$

Let $\Lambda(m,n)$ denote the lattice with basis m,n. Since v is primitive, we have that

$$\Lambda(m,n) = S(m,n) \cap \mathbb{Z}^{k+1},$$

where $S(m,n)$ denotes the subspace of $\mathbb{Q}^{k+1}$ spanned by the vectors m,n.

Consider the lattice $\Lambda^{\perp}(m,n)$ of integer vectors orthogonal to $S(m,n)$ and choose a basis

$$\begin{aligned} a'_1 &= (a_{11},\dots,a_{1k},a_{1k+1}), \\ &\vdots \\ a'_{k-1} &= (a_{k-11},\dots,a_{k-1k},a_{k-1k+1}), \\ a'_k &= (a_{k1},\dots,a_{kk},a_{kk+1}) \end{aligned} \tag{12}$$

of the lattice $\Lambda^{\perp}(n)$ such that the first $k-1$ vectors $a'_1 \dots a'_{k-1}$ form a basis of $\Lambda^{\perp}(m,n)$. It is easy to see that the vectors

$$\begin{aligned} a_1 &= (a_{11}, \dots, a_{1k}), \\ &\vdots \\ a_k &= (a_{k1}, \dots, a_{kk}) \end{aligned}$$

form a basis of $\Lambda^{\perp}_{k+1}(n)$. Consider the matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} & a_{1k+1} \\ & \ddots & & \\ a_{k-11} & \dots & a_{k-1k} & a_{k-1k+1} \end{pmatrix}$$

and denote by A_{ij} the minor obtained by omitting the ith and jth columns in A.

Let

$$V'_i = m_i n - n_i m$$

and let V_i be the vector obtained by omitting the ith entry in V'_i (note this entry is 0). When omitting the ith entry, we preserve the numbering of the remaining entries. For example, we consider V_3 as a vector of the k–dimensional space with coordinates x_1, x_2, x_4, $\dots$, x_{k+1}. In particular, $V_{k+1} = V$. Let $\Lambda^{\perp}_i(m,n)$ denote the lattice obtained by omitting the ith entries of all vectors of the lattice $\Lambda^{\perp}(m,n)$, preserving the numbering of the remaining entries. Denote by V_{ij} the jth entry of V_i. Then the following result holds.

Lemma 3.1. *$V_{ij} = \epsilon_{ij} A_{ij}$, where $\epsilon_{ij} = \pm 1$ and $\epsilon_{k+1i}\epsilon_{k+1j} = (-1)^{i-j}$.*

Proof. $V'_i \in \Lambda(m,n)$ implies that $V'_i \perp \Lambda^{\perp}(m,n)$ and thus $V_i \perp \Lambda^{\perp}_i(m,n)$. Hence V_i can be represented in the form

$$V_i = s_i(\text{external product of the vectors of a basis of } \Lambda^{\perp}_i(m,n)), \quad s_i \in \mathbb{R}. \tag{13}$$

Therefore,

$$V_{ij} = \epsilon_{ij} t_i A_{ij}, \ \epsilon_{ij} = \pm 1, \ t_i > 0$$

and clearly $\epsilon_{k+1i}\epsilon_{k+1j} = (-1)^{i-j}$. In order to see this, it is enough to note that the basis $\{a'_1 \dots a'_{k-1}\}$ of $\Lambda_i^{\perp}(m,n)$ obtained from (12) is a basis of the lattice on the right hand side of (13). Further, the equation $V_{ij} = -V_{ji}$ implies that $t_i = t_j$. Let $t = t_1 = \dots = t_k$. It is well known that

$$\det \Lambda(m,n) = \det \Lambda^{\perp}(m,n), \tag{14}$$

see e.g. [5], p. 27/28. For the first determinant holds

$$\det \Lambda(m,n) = \begin{pmatrix} mm & mn \\ mn & nn \end{pmatrix} = \frac{1}{2}\sum_{i\neq j} V_{ij}^2 = \frac{t^2}{2}\sum_{i\neq j} A_{ij}^2.$$

On the other hand, by the Laplace identity (see e.g. [16], Lemma 6D), we can write the second determinant as

$$\det \Lambda^{\perp}(m,n) = \det(a'_i a'_j)_{i,j=1}^{k-1} = \frac{1}{2}\sum_{i\neq j} A_{ij}^2,$$

and by (14) $t = t_1 = \dots = t_k = 1$. □

Since $V = n_{k+1}v$, Lemma 3.1 implies that the vector v is orthogonal to the vectors $a_1, \dots, a_{k-1}$ and

$$\begin{aligned} va_k &= \tfrac{1}{n_{k+1}} Va_k = \tfrac{1}{n_{k+1}}(V_{k+11}a_{k1} + \dots + V_{k+1k}a_{kk}) \\ &= \pm\tfrac{1}{n_{k+1}}(A_{k+11}a_{k1} - A_{k+12}a_{k2} + \dots + (-1)^{k-1}A_{k+1k}a_{kk}) \\ &= \pm\tfrac{1}{n_{k+1}} \det \Lambda_{k+1}^{\perp}(n) = \pm 1. \end{aligned}$$

By taking, if necessary, $-v$ instead of v, we may assume that $v\,a_k = 1$. This shows that $v \in \Lambda_{k+1}^{\perp}(n)^*$. Thus $\Lambda(n)$ is a sublattice of $\Lambda_{k+1}^{\perp}(n)^*$. Since

$$\det \Lambda(n) = \det(\Lambda_{k+1}^{\perp}(n))^* = \frac{1}{n_{k+1}},$$

these lattices coincide.

4. Proof of Theorem 1.1

Let $\{b_1^*, \dots, b_k^*\}$ be the basis of the polar lattice Λ^* given by

$$b_i^* b_j = \begin{cases} 1, & i = j\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

We shall apply Theorem 1 of [15], where $m = 1$, $F = 1$, and $F_{1\nu}$, $\nu = 1, \dots, k+1$ are the minors of order k of the matrix

$$\begin{aligned} M &= M(T, T_1, \dots, T_k) \\ &= \begin{pmatrix} db_{11}^*T + T_1 & db_{12}^*T & \dots & db_{1k}^*T & d\sum_{i=1}^k \alpha_i b_{1i}^*T \\ db_{21}^*T & db_{22}^*T + T_2 & \dots & db_{2k}^*T & d\sum_{i=1}^k \alpha_i b_{2i}^*T \\ \vdots & \vdots & & \vdots & \vdots \\ db_{k1}^*T & db_{k2}^*T & \dots & db_{kk}^*T + T_k & d\sum_{i=1}^k \alpha_i b_{ki}^*T \end{pmatrix}, \end{aligned}$$

where $T, T_1, \dots, T_k$ are variables. Let $M_i = M_i(T, T_1, \dots, T_k)$ and let B_i^* be the minor obtained by omitting the ith column in M or in the matrix

$$\begin{pmatrix} b_{11}^* & b_{12}^* & \dots & b_{1k}^* & \sum_{i=1}^k \alpha_i b_{1i}^* \\ b_{21}^* & b_{22}^* & \dots & b_{2k}^* & \sum_{i=1}^k \alpha_i b_{2i}^* \\ \vdots & \vdots & & \vdots & \vdots \\ b_{k1}^* & b_{k2}^* & \dots & b_{kk}^* & \sum_{i=1}^k \alpha_i b_{ki}^* \end{pmatrix},$$

respectively. As in the proof of Theorem 2 in [15] we have that

(15) $\quad |B_{k+1}^*| = |\det(b_{ij}^*)| \neq 0,$

(16) $\quad |B_i^*| = \alpha_i |B_{k+1}^*|,$

(17) $\quad M_i = d^k B_i^* T^k +$ polynomial of degree less than k in T

and $M_1, \dots, M_k$ have no common factor. By Theorem 1 of [15] there exist integers $t_1, \dots, t_k$ and an arithmetic progression $\mathcal{P}$ such that, for $t \in \mathcal{P}$, we have

$$\gcd(M_1(t, t_1, \dots, t_k), \dots, M_{k+1}(t, t_1, \dots, t_k)) = 1.$$

Let

$$n(t) = (M_1(t, t_1, \dots, t_k), \dots, (-1)^k M_{k+1}(t, t_1, \dots, t_k)).$$

Then (3)and (4) hold.

To prove the equality (2), consider the lattice $\Lambda_{k+1}^\perp(n(t))$, $t \in \mathcal{P}$ with basis

$$\begin{aligned} a_1^*(t) &= (db_{11}^* t + t_1, db_{12}^* t, \dots, db_{1k}^* t), \\ a_2^*(t) &= (db_{21}^* t, db_{22}^* t + t_2, \dots, db_{2k}^* t), \\ &\vdots \\ a_k^*(t) &= (db_{k1}^* t, db_{k2}^* t, \dots, db_{kk}^* t + t_k). \end{aligned}$$

By Lemma 2.1, $\Lambda(n(t))$ is the polar lattice of the lattice $\Lambda_{k+1}^\perp(n(t))$. Let $\{a_1(t), \dots, a_k(t)\}$ be a basis of $\Lambda(n(t))$ such that

$$a_i^*(t) a_j(t) = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}. \end{cases}$$

Consider the matrices $A^*(t) = (a_{ij}^*(t))_{i,j=1}^k$ and $B^* = (b_{ij}^*)_{i,j=1}^k$. Let $A_{ij}^*(t)$ and B_{ij}^* be the minors obtained by omitting the ith row and jth column in $A^*(t)$ and B^*, respectively. Then, in particular,

(18) $\quad A_{ij}^*(t) = d^{k-1} t^{k-1} B_{ij}^* + O(t^{k-2}).$

Moreover,

$$a_i(t) = \lambda^* (A_{i1}^*(t), -A_{i2}^*(t) \dots, (-1)^{k-1} A_{ik}^*(t)),$$

where $\lambda^* = \det \Lambda(n(t)) = (\det \Lambda_{k+1}^\perp(n(t)))^{-1}$. To check this, note that

$$\det \Lambda_{k+1}^\perp(n(t)) = a_i^*(t)(A_{i1}^*(t), -A_{i2}^*(t) \dots, (-1)^{k-1} A_{ik}^*(t)).$$

Analogously,

$$b_i = \lambda(B_{i1}^*, -B_{i2}^*, \dots, (-1)^{k-1} B_{ik}^*),$$

where $\lambda = (B^*_{k+1})^{-1} = (\det B^*)^{-1}$, since clearly

$$\det B^* = b^*_i(B^*_{i1}, -B^*_{i2} \dots, (-1)^{k-1} B^*_{ik}).$$

By (17)

$$\lambda^* = (d^k t^k \lambda^{-1} + O(t^{k-1}))^{-1}.$$

Thus by (18),

$$a_{ij}(t) = (-1)^{j-1} \frac{d^{k-1}t^{k-1}B^*_{ij} + O(t^{k-2})}{d^k t^k \lambda^{-1} + O(t^{k-1})} = (-1)^{j-1} \frac{d^{k-1}t^{k-1}B^*_{ij}}{d^k t^k \lambda^{-1}(1+O\left(\frac{1}{t}\right))} + O\left(\frac{1}{t^2}\right)$$
$$= (-1)^{j-1} \frac{\lambda B^*_{ij}}{dt} + O\left(\frac{1}{t^2}\right) = \frac{b_{ij}}{dt} + O\left(\frac{1}{t^2}\right).$$

5. Proof of Lemma 1.3

The functions f_t, f all are positive homogeneous of degree 1. Hence $f_t \to f$ uniformly on any bounded set. Thus

(19) $\quad l_t \to l$ implies $f_t(l_t) \to f(l)$ as $t \to \infty$.

(i): Let $\epsilon > 0$. Choose linearly independent vectors $l_1, \dots, l_k \in L$ such that

(20) $\quad \max\{f(l_1), \dots, f(l_i)\} \leq \lambda_i(f, L) + \epsilon$ for $i = 1, \dots, k$.

By Theorem 1 of [8], pp. 178–179, there exist vectors $l_{t1}, \dots, l_{tk} \in L_t$ such that

(21) $\quad l_{tj} \to l_j$ as $t \to \infty$ for $j = 1, \dots, k$.

Clearly,

(22) $\quad l_{t1}, \dots, l_{tk}$ are linearly independent for all sufficiently large t.

Thus

$$\lambda_i(f_t, L_t) \leq \max\{f_t(l_{t1}), \dots, f_t(l_{ti})\} \leq \max\{f(l_1), \dots, f(l_i)\} + \epsilon$$
$$\leq \lambda_i(f, L) + 2\epsilon \text{ for } i = 1, \dots, k \text{ and all sufficiently large } t$$

by (22), (21), (19) and (20), concluding the proof of (i).

(ii): Let $0 < \epsilon < 1$. Since $f_t \to f$ uniformly for $||x|| = 1$, $f(x) > 0$ for $||x|| = 1$ and f_t and f all are positive homogeneous of degree 1, there is an $\alpha > 0$ such that

(23) $\quad \alpha||x|| \leq (1-\epsilon)f(x) \leq f_t(x)$ for all x and all sufficiently large t.

For such t the function $f_t(x)$ is positive for $x \neq o$. Thus the star body $\{x : f_t(x) \leq 1\}$ is bounded. Hence we may choose

(24) $\quad l_{t1}, \dots, l_{tk} \in L_t$, linearly independent, such that

$\max\{f_t(l_{t1}), \dots, f_t(l_{ti})\} = \lambda_i(f_t, L_t), i = 1, \dots, k$ for all sufficiently large t.

By (23), (24) and (i),

(25) $$||l_{tj}|| \leq \frac{1}{\alpha} f_t(l_{tj}) \leq \frac{1}{\alpha} \lambda_i(f_t, L_t) \leq \frac{1}{\alpha} \lambda_d(f_t, L_t)$$
$$\leq \tfrac{1}{\alpha} \lambda_d(f, L) + \epsilon, \ j = 1, \dots, k \text{ for all sufficiently large } t.$$

Moreover,

$$|\det(l_{t1},\dots,l_{tk})| \geq \det L_t \geq \det L(1-\epsilon) \text{ for all sufficiently large } t$$

by (24) and since $L_t \to L$ and thus $\det L_t \to \det L$.

The sequences $(l_{t1}),\dots,(l_{tk})$ all are bounded by (25). The Bolzano – Weierstrass theorem thus shows that by considering suitable subsequences and re–indexing, if necessary, we may assume that

$$l_{tj} \to l_j \in L,\ |\det(l_1,\dots,l_k)| \geq \det L(1-\epsilon) > 0\,,$$

see [8], pp. 178–179, Theorem 1. Hence $l_1,\dots,l_k$ are linearly independent and $f_t(l_{tj}) \to f(l_j)$ by (19). Thus

$$\begin{aligned}&\lambda_i(f_t, L_t) = \max\{f_t(l_{t1}),\dots,f_t(l_{ti})\} \to \max\{f(l_1),\dots,f(l_i)\}\\ &\geq \lambda_i(f,L),\ i=1,\dots,k.\end{aligned}$$

Noting (i), this concludes the proof of (ii).

6. Proof of Theorem 1.2

The inequality

$$C(K) = \sup_{n\in\mathbb{U}^{k+1}} \frac{\lambda_1(K,\Lambda(n))\cdots\lambda_k(K,\Lambda(n))}{\det\Lambda(n)} \leq \frac{\alpha(K)}{\Delta(K)}$$

holds by the definition of anomaly (see [8], pp. 191, 192). To show that equality holds, it is sufficient to prove that

$$\sup_{n\in\mathbb{U}^{k+1}} \frac{\lambda_1(K,\Lambda(n))\cdots\lambda_k(K,\Lambda(n))}{\det\Lambda(n)} \geq \frac{\alpha(K)}{\Delta(K)}. \tag{26}$$

Let $\Lambda_0 = \Lambda_0(K)$ be a lattice such that

$$\lambda_1(K,\Lambda_0)\cdots\lambda_k(K,\Lambda_0) = \tfrac{\alpha(K)}{\Delta(K)}\det\Lambda_0. \tag{27}$$

The existence of such lattices for bounded star bodies in $\mathbb{R}^2$ was proved in [13] and for all dimensions in [9], see also [17]. Let $\{r_1,\dots,r_k\}$ be a basis of Λ_0. Let $0<\delta<1$ and choose linearly independent vectors $b_1(\delta),\dots,b_k(\delta)\in\mathbb{Q}^k$ such that

$$\begin{aligned}&||b_j(\delta)-r_j||_\infty < \delta,\ j=1,\dots,k,\\ &|\det(b_1^T(\delta),\dots,b_k^T(\delta)) - \det\Lambda_0| < \delta\det\Lambda_0.\end{aligned} \tag{28}$$

Apply Theorem 1.1 to the lattice Λ with basis $\{b_1(\delta),\dots,b_k(\delta)\}$ and arbitrarily chosen rational numbers $\alpha_1,\dots,\alpha_k$ with $0<\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_k\leq 1$. This gives an arithmetic progression $\mathcal{P}$ and a sequence $n(t) = (n_1(t),\dots,n_k(t),n_{k+1}(t)) \in \mathbb{Z}^{k+1}$, $t\in\mathcal{P}$, such that $\Lambda(n(t))$ has a basis $\{a_1(t),\dots,a_k(t)\}$ where

$$dta_{ij}(t) = b_{ij}(\delta) + O\left(\frac{1}{t}\right),\ i,j=1,\dots,k.$$

Here $d = d(\delta) \in \mathbb{N}$ such that $db_{ij}(\delta), d\alpha_j b_{ij}(\delta) \in \mathbb{Z}$ for all $i, j = 1, \ldots, k$. Choose any $t_0 = t_0(\delta) \in \mathcal{P}$ such that

(29) $\quad ||dt_0 a_j(\delta) - r_j||_\infty < \delta,\ j = 1, \ldots, k$

and $t_0 > 1/\delta$. Put $\Lambda_\delta = dt_0\Lambda(n(t_0))$. For $\delta \to 0$ we obtain an infinite sequence of lattices $\{\Lambda_\delta\}$ and by (29) $\Lambda_\delta \to \Lambda_0$. In view of Corollary 1.4,

$$\lambda_1(K, \Lambda_\delta) \cdots \lambda_k(K, \Lambda_\delta) \to \frac{\alpha(K)}{\Delta(K)} \det \Lambda_0 \text{ as } \delta \to 0.$$

We have

$$\lambda_1(K, \Lambda(n(t_0))) \cdots \lambda_k(K, \Lambda(n(t_0))) = \frac{\lambda_1(K, \Lambda_\delta) \cdots \lambda_k(K, \Lambda_\delta)}{(d(\delta)t_0(\delta))^k},$$

and by (3) and (28)

$$(d(\delta)t_0(\delta))^k = \frac{\det \Lambda}{\det \Lambda(n(t_0))} + O(t_0^{k-1}) < \frac{(1+\delta)\det \Lambda_0}{\det \Lambda(n(t_0))}(1 + O(\delta)).$$

Thus, for every $\epsilon > 0$ and for sufficiently small $\delta > 0$ there is an integer vector $n = n(t_0(\delta))$ such that

$$\lambda_1(K, \Lambda(n)) \cdots \lambda_k(K, \Lambda(n)) > \frac{(1-\epsilon)\alpha(K)}{\Delta(K)} \det \Lambda(n).$$

This implies (26) and shows that (5) holds for the sequence $\{n(t_0(\delta))\}$. For this sequence equality (6) holds by (4) and (7) holds by (3).

7. Proof of Theorem 1.5

We shall show that for every $\epsilon > 0$ there exists a vector $x \in \mathbb{R}^k$ and a real number $Q > 0$ such that

(30) $\quad \{\lambda_1(x, Q)\}^k > \frac{1-\epsilon}{\Delta(K)Q}.$

Let

$$C_1(K) := \limsup_{\substack{n \in \mathbb{U}^{k+1} \\ ||n||_\infty \to \infty}} \frac{\{\lambda_1(K, \Lambda(n))\}^k}{\det \Lambda(n)}.$$

The proof of Theorem 1.2 can be easily modified to prove that

(31) $\quad C_1(K) = \frac{1}{\Delta(K)}.$

We just have to take for the lattice $\Lambda_0 = \Lambda_0(K)$ any critical lattice of K and to replace (27) by the equality

$$\{\lambda_1(K, \Lambda_0)\}^k = \frac{\det \Lambda_0}{\Delta(K)}.$$

By (31) there is a sequence $\{n(t)\}$, such that $||n(t)||_\infty \to \infty$ and for all sufficiently large t holds

$$\{\lambda_1(K, \Lambda(n(t)))\}^k > \frac{(1-\epsilon)\det\Lambda(n(t))}{\Delta(K)\left(1 - \frac{1}{n_{k+1}(t)}\right)} = \frac{1-\epsilon}{\Delta(K)(n_{k+1}(t)-1)}.$$

Now put $x = (n_1(t)/n_{k+1}(t), \ldots, n_k(t)/n_{k+1}(t))$, $Q = n_{k+1}(t) - 1$ and note that $\lambda_1(K, \Lambda(n(t))) = \lambda_1(x, Q)$.

Remark. The proof of Theorem 1.5 does not yield only rational solutions x of the inequality (30) for $\epsilon > 0$. In fact, all vectors which are sufficiently close to a vector x satisfying (30) satisfy (30) as well. Moreover, since we apply Theorem 1.1 with arbitrarily chosen rational numbers α_i, the equality (4) implies that solutions of (30) approximate any rational point $(\alpha_1, \ldots, \alpha_k)$ with $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq 1$.

8. Proof of Theorem 1.6

For any $\epsilon > 0$ we have to find a sequence $\{n(t)\}$ of integer vectors such that $h(n(t)) \to \infty$ and for all sufficiently large t the following inequality holds:

$$\inf_{\substack{p,q\in\mathbb{Z}^{k+1} \\ \dim(p,q)=2 \\ n(t)=up+vq,u,v\in\mathbb{Z}}} \frac{h(p)h(q)}{h(n(t))^{1-\frac{1}{k}}} > \frac{1-\epsilon}{(k+1)^{\frac{1}{k}}}. \tag{32}$$

Let $n = (n_1, \ldots, n_{k+1})$, $0 < n_1 \leq \ldots \leq n_{k+1}$, be a primitive integer vector, that is $\gcd(n_1, \ldots, n_{k+1}) = 1$, and let $m = (m_1, \ldots, m_{k+1})$ be an integer vector, such that m and n are linearly independent. Consider the polygon $\Pi = \Pi(m, n)$ defined by

$$\Pi = \{(x, y) : |m_i y - n_i x| \leq 1 \text{ for } i = 1, \ldots, k+1\}. \tag{33}$$

Let

$$v = v(m) := \left(m_1 - m_{k+1}\tfrac{n_1}{n_{k+1}}, \ldots, m_k - m_{k+1}\tfrac{n_k}{n_{k+1}}\right) \in \Lambda(n). \tag{34}$$

The following lemma is implicit in [2].

Lemma 8.1. *Let $0 < n_1 < \ldots < n_{k+1}$ and $\xi > 0$. Then there is a centrally symmetric convex set $\mathcal{M}_\xi = \mathcal{M}_\xi(n) \subset \mathbb{R}^k$, such that $v(m) \in \mathcal{M}_\xi$ for an integer vector m if and only if*

$$\Delta(\Pi(m, n)) \geq \frac{1}{n_{k+1}\xi}.$$

Moreover,

$$V_k(\mathcal{M}_\xi) > (k+1)\xi^k. \tag{35}$$

Indeed, a set $\mathcal{M}_\xi$ satisfying the equivalence stated in Lemma 8.1 is described by formula (6) of [2] and the inequality (35) is proved in Lemma 12 ibid. Let $f_n(\cdot)$ be the distance function of the set $\mathcal{M}_1(n)$. By the definition of $\mathcal{M}_\xi$, for v as in (34), we have that

$$f_n(v) = (n_{k+1}\Delta(\Pi))^{-1}. \tag{36}$$

Consider a generalized honeycomb E_1^k given by the inequalities

$$E_1^k = \{x \in \mathbb{R}^k : |x_i| \le 1, |x_i - x_j| \le 1 \text{ for } i, j = 1, \dots, k, i \ne j\}.$$

Observe that

$$E_1^k = \bigcap_{p<q} \left\{x \in \mathbb{R}^k : (x_p, x_q) \in E_1^2\right\}.$$

Let $g_k(\cdot)$ be the distance function of E_1^k. Then clearly

$$g_k(x) = \max_{1\le i<j\le k} g_2((x_i, x_j)).$$

By Lemma 1 of [2],

$$V_k(E_1^k) = k+1, \Delta(E_1^k) = \frac{k+1}{2^k}$$

and E_1^k has a unique critical lattice $\Lambda(E_1^k)$ with basis

$$\begin{aligned} b_1 &= (1, 1/2, \dots, 1/2), \\ b_2 &= (1/2, 1, \dots, 1/2), \\ &\vdots \\ b_k &= (1/2, 1/2, \dots, 1). \end{aligned}$$

Lemma 8.2. *For any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for all integer vectors $n = (n_1, \dots, n_k, n_{k+1})$ with $1 - \delta < n_1/n_{k+1} < \dots < n_k/n_{k+1} < 1$, for all $x \in \mathbb{R}^k \setminus \{o\}$*

$$f_n(x) > \left(1 - \frac{\epsilon}{2}\right) g_k(x).$$

Proof. By formula (6) of [2], the set $\mathcal{M}_1(n)$ is the intersection of the sets $\mathcal{G}_{pqr}$, where

$$\mathcal{G}_{pqr} = \left\{x \in \mathbb{R}^k : (x_p, x_q) \in \mathcal{B}_1\left(\frac{n_p}{n_{k+1}}, \frac{n_q}{n_{k+1}}\right)\right\}$$

for $p < q < r = k+1$ and

$$\mathcal{G}_{pqr} = \left\{x \in \mathbb{R}^k : \left(x_p - \frac{n_p}{n_r}x_r, x_q - \frac{n_q}{n_r}x_r\right) \in \gamma\mathcal{B}_1\left(\frac{n_p}{n_r}, \frac{n_q}{n_r}\right)\right\}$$

for $p < q < r < k+1$, $\gamma = n_{k+1}/n_r$. The set $B_1 = B_1(\alpha, \beta)$, $0 < \alpha < \beta < 1$ is defined by the formulae (8)–(13) of [1]. The boundary of $\mathcal{B}_1$ consists of two horizontal segments

$$\pm S_h = \left\{\pm(t, 1) \in \mathbb{R}^2 : -\frac{1-\alpha}{1+\beta} \le t \le \frac{1+\alpha}{1+\beta}\right\},$$

two vertical segments

$$\pm S_v = \left\{\pm(1, t) \in \mathbb{R}^2 : -\frac{1-\beta}{1+\alpha} \le t \le \frac{1-\beta}{1-\alpha}\right\},$$

and four curvilinear arcs $\pm L_1$, $\pm L_2$ with

$$\pm L_1 = \left\{ \pm(x(t), tx(t)) \in \mathbb{R}^2 : \frac{1-\beta}{1-\alpha} \leq t \leq \frac{1+\beta}{1+\alpha} \right\},$$
$$x(t) = \frac{-t^2(1+\alpha)^2 + 2t(1-\alpha+\beta+\alpha\beta) - (1-\beta)^2}{4t(\beta-\alpha t)}$$

and

$$\pm L_2 = \left\{ \pm(X(t), -tX(t)) \in \mathbb{R}^2 : \tfrac{1-\beta}{1+\alpha} \leq t \leq \tfrac{1+\beta}{1-\alpha} \right\},$$
$$X(t) = \frac{-t^2(1-\alpha)^2 + 2t(1+\alpha+\beta-\alpha\beta) - (1-\beta)^2}{4t(\beta+\alpha t)}.$$

By Lemma 1 of [1], B_1 is a centrally symmetric convex set.

Assume that there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exist an integer vector $n = (n_1, \ldots, n_k, n_{k+1})$ with $1 - \delta < n_1/n_{k+1} < \ldots < n_k/n_{k+1} < 1$ and a point $x \in \mathbb{R}^k \setminus \{o\}$ with

$$f_n(x) \leq \left(1 - \tfrac{\epsilon}{2}\right) g_k(x). \tag{37}$$

We shall show that this leads to a contradiction. By (37), there is a point $a = (a_1, \ldots, a_k) = \lambda x$, $\lambda > 0$, such that $f_n(a) = 1$ and

$$g_k(a) = g_2((a_i, a_j)) \geq \left(1 - \tfrac{\epsilon}{2}\right)^{-1} \tag{38}$$

for some $i, j = 1, \ldots, k$, $i < j$. Let $\alpha = n_i/n_{k+1}$, $\beta = n_j/n_{k+1}$. Since $a \in \mathcal{M}_1(n)$, we have $(a_i, a_j) \in B_1(\alpha, \beta)$.

First, we consider the case $a_i a_j \geq 0$. By Lemma 2 of [1]

$$B_1(\alpha, \beta) \subset C_1 := \{x \in \mathbb{R}^2 : ||x||_\infty \leq 1\} \tag{39}$$

and thus

$$\{(x_i, x_j) \in B_1(\alpha, \beta) : x_i x_j \geq 0\} \subset \{(x_i, x_j) \in E_1^2 : x_i x_j \geq 0\},$$

which contradicts (38).

Let us now consider the case $a_i a_j < 0$. Suppose $a_j = -t a_i$. We may assume without loss of generality that

$$\left(1 - \tfrac{\epsilon}{2}\right)^{-1} - 1 \leq t \leq \left(\left(1 - \tfrac{\epsilon}{2}\right)^{-1} - 1 \right)^{-1}. \tag{40}$$

Otherwise $(a_i, a_j) \notin C_1$ and we get a contradiction with (39). Since $(1 - \beta)/(1+\alpha)$ tends to 0 and $(1+\beta)/(1-\alpha)$ tends to infinity as δ tends to 0, we have

$$\frac{1-\beta}{1+\alpha} < t < \frac{1+\beta}{1-\alpha}$$

for δ small enough. Then $\mu(a_i, a_j) \in \pm L_2$ for some $\mu \geq 1$. Further, for any t from the interval (40)

$$X(t) \to \frac{1}{1+t}, \quad \text{as } \delta \to 0.$$

Since $g_2(1/(1+t), -t/(1+t)) = 1$, we obtain a contradiction with (38) for all sufficiently small δ. $\square$

Lemma 8.3. *For any $\epsilon > 0$, there is an arithmetic progression $\mathcal{P}$ and a sequence of primitive integer vectors $n(t) = (n_1(t), \ldots, n_k(t), n_{k+1}(t))$, $t \in \mathcal{P}$, such that $h(n(t)) \to \infty$ and for all sufficiently large $t \in \mathcal{P}$, for every non–zero vector $v \in \Lambda(n(t))$ the following holds*

$$f_{n(t)}(v) > (1-\epsilon)\left\{n_{k+1}(t)\Delta(E_1^k)\right\}^{-\frac{1}{k}}.$$

Proof. Choose rational numbers $1 - \delta(\epsilon) < \alpha_1 < \alpha_2 < \cdots < \alpha_k < 1$ and apply Theorem 1.1 to the lattice $\Lambda = \Lambda(E_1^k)$, the basis $\{b_1, \ldots, b_k\}$ of Λ and the numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$. This yields an arithmetic progression $\mathcal{P}$ and a sequence of primitive integer vectors $n(t)$, $t \in \mathcal{P}$ such that $h(n(t)) \to \infty$ and the corresponding lattices $\Lambda(n(t))$ have bases $a_1(t), \ldots, a_k(t)$ where

$$a_{ij}(t) = \frac{b_{ij}}{dt} + O\left(\frac{1}{t^2}\right),\ i, j = 1, \ldots, k. \tag{41}$$

Here $d \in \mathbb{N}$ is such that $db_{ij}, d\alpha_j b_{ij} \in \mathbb{Z}$ for all $i, j = 1, \ldots, k$. Moreover,

$$\alpha_i(t) := \frac{n_i(t)}{n_{k+1}(t)} = \alpha_i + O\left(\frac{1}{t}\right).$$

Thus for sufficiently large t,

$$1 - \delta(\epsilon) < \frac{n_1(t)}{n_{k+1}(t)} < \ldots < \frac{n_k(t)}{n_{k+1}(t)} < 1.$$

We now show that for sufficiently large $t \in \mathcal{P}$

$$\lambda_1(E_1^k, \Lambda(n(t))) > \left(1 - \frac{\epsilon}{2}\right)\{n_{k+1}(t)\Delta(E_1^k)\}^{-\frac{1}{k}}. \tag{42}$$

The equality (41) implies that

$$dt\Lambda(n(t)) \to \Lambda, \text{ as } t \to \infty,\ t \in \mathcal{P}.$$

Thus, Lemma 1.3 implies that

$$\lambda_1(E_1^k, dt\Lambda(n(t))) \to 1, \text{ as } t \to \infty, t \in \mathcal{P}.$$

Since

$$\lambda_1(E_1^k, \Lambda(n(t))) = \frac{\lambda_1(E_1^k, dt\Lambda(n(t)))}{dt}$$

and by (3),

$$dt = (n_{k+1}(t)\det\Lambda)^{\frac{1}{k}}\left(1 + O\left(\frac{1}{t}\right)\right)^{\frac{1}{k}},$$

the inequality (42) holds for all sufficiently large t. By Lemma 8.2 and (42) for sufficiently large $t \in \mathcal{P}$ for every non–zero vector $v \in \Lambda(n(t))$,

$$\begin{aligned} f_{n(t)}(v) &> \left(1 - \tfrac{\epsilon}{2}\right) g_k(v) \geq \left(1 - \tfrac{\epsilon}{2}\right)\lambda_1(E_1^k, \Lambda(n(t))) \\ &> (1-\epsilon)\{n_{k+1}(t)\Delta(E_1^k)\}^{-\frac{1}{k}}. \end{aligned}$$

The proof of the Lemma 8.3 is complete. □

After these preparations, the proof of Theorem 1.6 is rather simple. We shall show that for every $\epsilon > 0$ the sequence $\{n(t)\}_{t\in\mathcal{P}}$ obtained in Lemma 5 satisfies (32) for all sufficiently large t. Let $t \in \mathcal{P}$ and let $p, q \in \mathbb{Z}^{k+1}$ be linearly independent vectors such that $n(t) = up + vq$ with $u, v \in \mathbb{Z}$, that is $n(t) \in \Lambda(p, q)$. Since the vector $n(t)$ is primitive, it can be extended to a basis of the lattice $S(p, q) \cap \mathbb{Z}^{k+1}$ by an integer vector m. Consider the polygon $\Pi = \Pi(m, n(t))$ given by (33). By Minkowski's lower bound for the product of successive minima and since $V_2(\Pi) \leq 4\Delta(\Pi)$, for all linearly independent integer vectors (x_1, y_1) and (x_2, y_2),

$$\prod_{i=1}^{2} h(y_i m - x_i n(t)) \geq \lambda_1(\Pi, \mathbb{Z}^2)\lambda_2(\Pi, \mathbb{Z}^2) \geq 2(V_2(\Pi))^{-1} \geq \frac{1}{2}(\Delta(\Pi))^{-1}.$$

Since $p, q \in \Lambda(m, n(t))$, we have that

$$h(p)h(q) \geq \frac{1}{2}(\Delta(\Pi))^{-1}. \tag{43}$$

By (4), for all sufficiently large t we have $h(n(t)) = n_{k+1}(t)$. Finally, by (43), (36) and Lemma 8.3, for sufficiently large t, we get

$$\frac{h(p)h(q)}{h(n(t))^{1-\frac{1}{k}}} \geq \frac{1}{2}(n_{k+1}(t))^{\frac{1}{k}} f_{n(t)}(v(m)) > \frac{1-\epsilon}{(k+1)^{\frac{1}{k}}}.$$

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