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UNAVOIDABLE ARRAYS

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Abstract. An $n \times n$ array is *avoidable* if for each set of n symbols there is a Latin square on these symbols which differs from the array in every cell. We characterise all unavoidable square arrays with at most 2 symbols, and all unavoidable arrays of order at most 4. We also identify a number of general families of unavoidable arrays, which we conjecture to be a complete account of unavoidable arrays. Next, we investigate arrays with multiple entries in each cell, and identify a number of families of unavoidable multiple entry arrays. We also discuss fractional Latin squares, and their connections to unavoidable arrays.

We note that when rephrasing our results as edge list-colourings of complete bipartite graphs, we have a situation where the lists of available colours are shorter than the length guaranteed by Galvin's Theorem to allow proper colourings.

1. Introduction

An $n \times n$ *Latin square* is an $n \times n$ array on n symbols, usually taken to be $[n] = \{1, \ldots, n\}$, such that each symbol occurs exactly once in each row and each column. An array A is *avoided* by an array B of the same order if entries in corresponding cells are different. An array A is *avoidable* if for each set of n symbols, there is a Latin square on these symbols that avoids A. We allow for empty cells, and symbols other than $[n]$, though such additional symbols can be disregarded. A *multiple entry array* is an array where each cell can hold several symbols.

We say that two arrays are *isotopic* if one can be transformed into the other by suitable permutations of the rows, the columns and/or the symbols. In somewhat non-standard terminology, we shall also allow the columns and rows to switch roles, and still use the term 'isotopic'. Two arrays are *conjugate* if one can be transformed into the other by freely exchanging the roles of rows, columns and symbols, and permuting elements within these classes. Obviously, two isotopic arrays are conjugate. Any conjugate of a (partial) Latin square is a (partial) Latin square, so when investigating avoidable (multiple entry) arrays, we need really only consider distinct conjugacy classes.

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The question of which $n \times n$ arrays are avoidable was posed by Häggkvist in 1989 [8]. It is not hard to produce examples of unavoidable arrays, but no complete (nor even partial) characterisation is known. By results of Chetwynd and Rhodes [4], Cavenagh [2], and Cavenagh and one of the present authors [3], all partial Latin squares of order at least 4 are avoidable, and there exist unavoidable partial Latin squares of order 2 and 3. We may therefore restrict our search for unavoidable arrays to arrays that have at least one repeated entry in some row or column.

As is well known, proper edge colourings of complete balanced bipartite graphs correspond to Latin squares. Galvin's Theorem [7] determines a condition on how many colours must be available on each edge to ensure the existence of a proper edge colouring using only allowed colours at each edge of a general bipartite graph. In the edge colouring language, we investigate what conditions we must place on the lists of available colours if we want them to be slightly shorter than the length specified in Galvin's Theorem. This is a small special case of the more general interesting question of what conditions must be placed on the lists in a general bipartite multigraph in order to ensure colourability.

The present investigation is in the same spirit as [6], where Brooks' bound on the chromatic number, when odd cycles and Δ-cliques are excluded, is improved on to $\Delta - k$ for a range of k, by excluding more and more subgraphs. In our case, we find certain configurations of disallowed colours that make proper colouring impossible, but when these specific configurations are excluded, colouring is always possible.

We shall start by proving a characterisation of all unavoidable arrays on exactly one or two symbols. Next, we list all small (orders 2, 3 and 4) minimal unavoidable arrays, produced by exhaustive computer search. We then investigate the avoidability of multiple entry arrays. Finally, we state some conjectures as to the set of all unavoidable arrays.

2. Arrays with one or two distinct symbols

In what follows, a recurring configuration is an $r \times (n - r + 1)$ rectangle, which we name a *critical rectangle*, and *diagonals*, which are just a set of n cells, exactly one from each row and each column. We shall denote by $A[\sigma]$ the set of cells of A containing the symbol σ.

It is reasonable to expect that unavoidable arrays using few symbols are not very common. In the next few results, we characterise those using exactly one or exactly two distinct symbols. It is obvious that a critical rectangle completely filled with a single symbol makes for an unavoidable array. Any additional occurences of the symbol in the critical rectangle are not necessary, and can be removed, to achieve minimality. It would be conceivable, however, that there are other configurations using only one symbol that are also unavoidable, but the next proposition, which is basically just a corollary to Hall's Theorem, rules this out.

Proposition 2.1. *Any unavoidable $n \times n$ array A using only the symbol 1 contains a critical rectangle $R' \subset A[1]$.*

Proof. To produce a Latin square avoiding an array A with only the symbol 1, we need only find a diagonal of empty cells in the array. Here we can enter the symbol 1, and the completion to a Latin square is trivial, since no other symbols are prohibited. By Hall's Theorem, an array A using only one symbol has an empty diagonal if and only if for each set of rows R, the number of columns C that have at least one empty cell in one of the rows in R is at least $r = |R|$.

In other words, an unavoidable array using only one symbol must have, for some $r \leq n$, a set R of r rows, where the empty cells in these rows occur in at most $r-1$ distinct columns. Therefore, there is a set of at least $n-(r-1)$ columns having no 1s in the rows of R. The intersection of R with these columns gives us our critical rectangle. $\square$

When looking at unavoidable arrays with two distinct symbols, we find that we shall want to say when an avoidable array with one single symbol forces the use of that symbol in all of a set of cells or in at least one of a set of cells.

Lemma 2.2. *Let A be an $n \times n$ avoidable array using only the symbol 1. Suppose that any Latin square that avoids A must use the symbol 1 in each of the cells in the set S. Then for each cell $(i,j) \in S$, A contains a critical rectangle $R_{i,j} \ni (i,j)$, such that $R_{i,j} \setminus (i,j) \subset A[1]$.*

Proof. If we are forced to place the symbol 1 in cell (i,j) when avoiding A, it follows that adding a 1 to cell (i,j) results in an unavoidable array. By Proposition 2.1 the array now contains a critical rectangle on the symbol 1. Removing the added 1 from A results in the subarray claimed to exist. $\square$

Lemma 2.3. *Let A be an $n \times n$ avoidable array using only one symbol, 1. Suppose that any Latin square that avoids A must use the symbol 1 in at least one of the cells in the set of cells S. Then there is a nonempty subset $T \subset S$, such that A contains a critical rectangle $R \supset T$ such that $R \setminus T \subset A[1]$.*

Proof. We assume that there is no cell in S where the symbol 1 is forbidden, for then we simply remove it from S. We form T in the following way: if we add a forbidden 1 in each cell of S, we get an unavoidable array $A^\star$, which must therefore contain a critical rectangle $R \subset A^\star[1]$. R intersects S, for the removal of the added 1s in the cells of S would result in an avoidable array, and this can therefore not contain a critical rectangle on the symbol 1. If we set $T = R \cap S$ we see that R covers T, all cells of T are empty because all cells of S are, and $T \subset S$. $\square$

We are now in a position to prove a characterisation of unavoidable arrays on two distinct symbols.

Theorem 2.4. *Let A be an $n \times n$ unavoidable array with two distinct symbols, 1 and 2, that does not constitute an unavoidable array when either symbol is completely removed. Then A contains one $r \times (n-r+1)$ array R_1 and one $(n-r+1) \times r$ array R_2 as follows: R_1 and R_2 intersect in a single cell $\{c\}$ which is empty, and $R_i \setminus \{c\} \subset A[i]$ for $i = 1, 2$.*

Proof. Since we assume that neither the 1s nor the 2s constitute an unavoidable array in themselves, we can assume that we may place the 1s on a diagonal, respecting the constraints set by the 1s in A. Then, no matter how these 1s are placed, there is not enough room to place the 2s. If it were possible to place the 2s, we could easily fill in the rest of the Latin square, as there are no other symbols in A.

For $i = 1, 2$ let M_i be the set of cells where $A[i]$ forces us to use the symbol i, i.e. the set of cells that all diagonals that avoid $A[i]$ intersect. By Lemma 2.2, M_i consists only of cells $\{c\}$ such that $\{c\} \cup A[i]$ contains a critical rectangle. We shall prove that $M_1 \cap M_2 \neq \varnothing$, so that by Lemma 2.2, for each cell $\{c\} \in M_1 \cap M_2$ we find the critical rectangles claimed.

By Proposition 2.1, for any diagonal D with $D \cap A[1] = \varnothing$, $A[2] \cup D$ contains a critical rectangle, for otherwise we could find a suitable diagonal to place 2s in, which would contradict unavoidability.

Thus each diagonal $D \cap A[1] = \varnothing$ contributes to some critical rectangles in $A[2] \cup D$. Suppose there is some diagonal D_1 whose contribution to all critical rectangles in $A[2] \cup D_1$ is at least two cells.

Then we claim that we can form a new diagonal D_1' such that $D_1' \cup A[2]$ does not contain any critical rectangle. Indeed, we can, separately for each critical rectangle $R \subset D_1 \cup A[2]$, pick out two cells in $D_1 \cap R$ and reform D_1 so that R no longer lies in the union of $A[2]$ and the reformed diagonal. This process is illustrated in Figure 1, where we read "a/b" as a being forbidden, and b being used in that cell, and $\varnothing$ indicates an empty cell.

Since the change from D_1 to D_1' only involves moving two 1s into $A[2]$, this reformation produces no new critical rectangles, so after a number of such single reformations, we find a diagonal D_1' such that $D_1' \cup A[2]$ contains no critical rectangles. Thus there is some critical rectangle to which D_1 only contributes one cell, $\{c\}$, so that $\{c\} \in M_2$. Thus M_2, and by symmetry, M_1 are non-empty.

We now prove that any diagonal D_1 that avoids $A[1]$ intersects M_2 and vice versa. If there were a diagonal D_2 avoiding $A[2] \cup D_1$, we could fill D_i with symbol i and easily complete this to a Latin square, contradicting the unavoidability of A. Therefore, $A[2] \cup D_1$ contains critical rectangles R. Suppose now that $D_1 \cap M_2 = \varnothing$. Then D_1 contributes at least 2 cells to each critical rectangle in $A[1] \cup D_1$. If this were the case, we could, as in Figure 1, reform D_1 to find a new diagonal D_1' such that $A[1] \cup D_1'$ contains no critical rectangles, which would contradict the unavoidability of A.

Since every diagonal $D_1 \cap A[1]$ intersects M_2, by Lemma 2.3, there is a nonempty set $T \subset M_2$ such that $A[1] \cup T$ contains a critical rectangle.

$$\begin{array}{|c|c|c|} \hline \varnothing/1 & \cdots & 2/\varnothing \\ \hline \cdots & & \cdots \\ \hline 2/\varnothing & \cdots & \varnothing/1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline \varnothing/\varnothing & \cdots & 2/1 \\ \hline \cdots & & \cdots \\ \hline 2/1 & \cdots & \varnothing/\varnothing \\ \hline \end{array}$$

Figure 1: Forming D_1'.

We now consider $|T|$. If $|T| = 1$, it is obvious that $T \subset M_1$. If it were the case that $|T| \geq 2$, we would not be forced to use symbol 2 in any of the cells of T, contradicting the fact that $T \subset M_2$. To see why this is so, we take two distinct cells in T. Figure 2 shows how the 2s claimed to be forced in the these two cells of T can be moved.

$$\begin{array}{|c|c|c|} \hline 1/\varnothing & \cdots & \varnothing/2 \\ \hline \cdots & & \cdots \\ \hline \varnothing/2 & \cdots & 1/\varnothing \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 1/2 & \cdots & \varnothing/\varnothing \\ \hline \cdots & & \cdots \\ \hline \varnothing/\varnothing & \cdots & 1/2 \\ \hline \end{array}$$

Figure 2: Avoiding 2 cells of T.

Thus $|T| = 1$ and therefore $T \subset M_1$, so that $T \subset M_1 \cap M_2$. This completes the proof. ☐

We now properly define arrays of type $A_{n,r}$ and $B_{n,r}$, and note again that we have a complete characterisation of unavoidable arrays on one or two symbols. We state the characterisation as a theorem.

Definition 2.5. *We denote by $A_{n,r}$ the $n \times n$ unavoidable 1-symbol array that has an $r \times (n - r + 1)$ subarray filled with that symbol.*

By $B_{n,r}$ we denote the $n \times n$ unavoidable 2-symbol array that has an $r \times (n - r + 1)$ subarray R_1 filled with 1s except for one cell c, and an $(n - r + 1) \times r$ subarray R_2 filled with 2s except for the cell $c = R_1 \cap R_2$.

Theorem 2.6. *Let M be a minimal $n \times n$ unavoidable array on one or two symbols. Then $M \in \{A_{n,1}, \ldots, A_{n,\lceil \frac{n}{2} \rceil}, B_{n,1}, B_{n,3}, \ldots, B_{n,\lceil \frac{n}{2} \rceil}\}$.*

Unavoidable arrays with exactly one symbol are easily grasped. In Figure 3, we see, however, that $B_{5,2}$ is not minimal, for it contains $B_{5,1}$ as a subarray, and that $B_{5,3}$ is minimal. Theorem 2.4 states that the minimal unavoidable arrays with 2 distinct symbols are exactly the arrays of type B, with the exception of $B_{n,2}$ for all n.

1				
1				
1				
1				
	2	2	2	2

(a) $B_{5,1}$

1	1			
1	1			
1	1			
1		2	2	2
	2	2	2	2

(b) $B_{5,2}$

1	1	1		
1	1	1		
1	1		2	2
		2	2	2
		2	2	2

(c) $B_{5,3}$

Figure 3: All 5×5 arrays of type B.

We conclude this section by, for future reference, recording in the following lemma when a 2-symbol array forces the use of one of these symbols in all of a specified set of *empty* cells, and remark that the general case when the cells in which a 1 or 2 is forced are not empty is considerably harder to make sense of.

Lemma 2.7. *Let A be an $n \times n$ avoidable array on symbols 1 and 2, and suppose that cell c is empty and any Latin square that avoids A uses a 1 in cell c. Then one of the following holds.*

(a) *There is an critical rectangle R_1 covering c, with $R_1 \setminus \{c\} \subset A[1]$.*
(b) *There are two critical rectangles R_1 and R_2, $c \in R_1$, $R_1 \cap R_2 = e$, cell e is empty, $e \neq c$, $R_1 \setminus \{c, e\} \subset A[1]$, and $R_2 \setminus \{e\} \subset A[2]$.*

Proof. Placing an additional forbidden 1 in cell c produces an unavoidable array, $A^\star$. If $A^\star$ is unavoidable on account of the 1s alone, we have Case (a), by Proposition 2.1. If both 1s and 2s play a part in making $A^\star$ unavoidable, then by Theorem 2.4 we have Case (b). $\square$

3. Investigating small arrays

By Proposition 2.1 and Theorem 2.4, the only minimal unavoidable arrays on one (labelled A with indexing subscripts) or two symbols (labelled B with indexing subscripts) for $2 \le n \le 4$ are, up to isotopism, those given in Figure 4. Note that these include both unavoidable 2×2 minimal unavoidable arrays.

Definition 3.1. *We denote by $D_{n,r}$ an $n \times n$ array where the first $n-1$ cells of the i:th column is filled with symbol i for $1 \le i \le r-1$, and the last $n-r+1$ cells in the last row are filled with symbol r.*

We denote by $C_{n,1}$ *the* $n \times n$ *array where the cells* $(1,1), \ldots, (n-1,1)$ *and* $(n-1,3), \ldots, (n-1,n-1)$ *are filled with 1s, the cells* $(1,2), \ldots, (n-2,2)$ *and* $(n,2), \ldots, (n,n-1)$ *are filled with 2s and the cells* $(1,n), \ldots, (n-1,n)$ *are filled with 3s.*

We denote by $C_{n,2}$ *the* $n \times n$ *array where the* $n-1$ *first cells in the first row and first column are filled with 1s, the cells* $(n,2), \ldots, (n,n-1)$ *are filled with 2s and the cells* $(2,n), \ldots, (n-1,n)$ *are filled with 3s.*

The label $C_{n,3}$ *is synonymous with* $D_{n,3}$*.*

1	1

(a) $A_{2,1}, D_{2,1}$

1	
	2

(b) $B_{2,1}, D_{2,2}$

1	1	1

(c) $A_{3,1}, D_{3,1}$

1	1	
1	1	

(d) $A_{3,2}$

1		
1		
	2	2

(e) $B_{3,1}, D_{3,2}$

1	1	1	1

(f) $A_{4,1}, D_{4,1}$

1	1	1	
1	1	1	

(g) $A_{4,2}$

1			
1			
1			
	2	2	2

(h) $B_{4,1}, D_{4,2}$

Figure 4: Minimal unavoidable arrays on one or two symbols.

All minimal unavoidable 3×3 arrays on 3 symbols (up to isotopism), generated by computer, are given in Figure 5. The minimal unavoidable non-isotopic 4×4 arrays on at least 3 symbols corresponding to these are presented in Figure 6. Unexpected additions to the list of 4×4 unavoidable arrays, all using 4 symbols, are given in Figure 7.

These are minimal in the sense that the removal of any entry results in an avoidable array. The list is complete in the sense that any unavoidable array contains one of them as a subarray. We will refer our unavoidable arrays as being of *type* A through D or S, as listed. Types A–D are evidently parts of larger families of unavoidable arrays, that have members for each order. That "type" S generalises to any order is not as clear. In fact, we conjecture that the arrays of type S presented here are exceptional examples, hence the label S for 'Sporadic'. This conjecture will be articulated more precisely below.

1	2	3
1		
	2	

(a) $C_{3,1}$

1	1	
1		3
	2	

(b) $C_{3,2}$

1	2	
1	2	
		3

(c) $C_{3,3}, D_{3,3}$

1	2	3
1	3	2

(d) $S_{3,1}$

1	2	3
3		1
2	1	

(e) $S_{3,2}$

Figure 5: All minimal unavoidable 3×3 arrays on 3 symbols.

In Figure 5 we find an old aquaintance from [8], namely $S_{3,1}$, which was identified as the only known example of an unavoidable row-latin array with empty last row.

1	2		3
1	2		3
1		1	
	2	2	

(a) $C_{4,1}$

1	1	1	
1			3
1			3
	2	2	

(b) $C_{4,2}$

1	2		
1	2		
1	2		
		3	3

(c) $C_{4,3}, D_{4,3}$

1	2	3	
1	2	3	
1	2	3	
			4

(d) $D_{4,4}$

1	1	2	3
1	1	3	2
		2	2
		3	3

(e) $S_{4,1}$

1	2	2	3
3	4	4	1
3	4	4	1
2	1	1	

(f) $S_{4,2}$

Figure 6: Corresponding minimal unavoidable 4×4 arrays.

The list of unavoidable 4×4 arrays in Figures 4 through 7 was generated by computer in the following manner.

First, the set of all avoidable arrays on the symbol 1 were generated, and reduced with respect to isomorphism using Brendan McKay's isomorphism

1	1		2
1	1		2
3	2	2	4
2	4	2	3

(a) $S_{4,3}$

1	1		2
1	1		2
3	3	2	4
4	4	2	3

(b) $S_{4,4}$

1	1		
1	1	2	2
2	3	2	4
4	2	2	3

(c) $S_{4,5}$

1	1		
1	1	2	2
3	3	2	4
4	4	2	3

(d) $S_{4,6}$

1	1	1	
1	1		2
2	3	2	4
4	2	2	3

(e) $S_{4,7}$

1	1	1	
1	1		2
3	3	2	4
4	4	2	3

(f) $S_{4,8}$

1	1	1	
1	3	4	3
2	2		2
2	3	3	4

(g) $S_{4,9}$

Figure 7: All minimal unavoidable 4×4 arrays not included in Figures 4 or 6.

package Nauty [10]. Next, for each of the generated 1-symbol arrays, forbidden 2s were added, in each case fewer than or equal to the number of already present 1s. Another run of Nauty reduced this list. In the same manner, symbols 3 and 4 were added. The final reduced list was then tested for avoidability and minimality using the SAT-solver UMSAT developed at Umeå University [13].

We note that among the arrays in this section, those labelled with $C_{n,1}$, $C_{n,2}$, $C_{n,3}$ and $D_{n,r}$ can be generalised to any n, and that we have found no general construction to extend any of the arrays of type S to arrays of general size. We phrase this as a proposition.

Proposition 3.2. *For $n \geq 3$ there exist at least three distinctive types of unavoidable arrays on three symbols: $C_{n,1}$, $C_{n,2}$ and $C_{n,3}$. Also, for any $r \leq n$ there exist unavoidable arrays on r distinct symbols, namely the $D_{n,r}$.*

4. Multiple entry arrays

In [5] a rough counting method yielded some general results on avoidable multiple entry arrays, but here we aim for precise descriptions of the border between avoidable and unavoidable.

4.1. **Entries in only one or two rows.** If we allow ourselves to forbid more than one symbol in each cell of an array, our list of unavoidable arrays will evidently grow longer, but when all entries are in the first two rows, we can, by taking conjugates, use Theorem 2.4 to obtain a full characterisation. We omit the details.

4.2. **Entries only on a diagonal.** Making use of the following characterisation by G. J. Chang, cited (and again proved) in [9], of completable partial Latin squares having all their entries on a diagonal, we can characterise all unavoidable multiple entry arrays with entries only on a diagonal.

Theorem 4.1. *Let D be an $n \times n$ array with entries on a diagonal. D is completable if and only if no symbol occurs exactly $n-1$ times.*

When applied to avoidability, Theorem 4.1 implies that if we exclude the trivial case of when all symbols are forbidden in some single cell, a multiple entry array with entries only on the diagonal is unavoidable exactly when it forces us to use one symbol exactly $n-1$ times, and a different symbol in the last cell on the diagonal. It may seem that a direct description of the unavoidable arrays presently considered is trivial, but it turns out that there is one twist to the story. It is fairly obvious that if symbols, say, $1, \ldots, (n-1)$ are all forbidden in the $n-1$ first cells along the diagonal, and symbol n is fobidden in the last cell, then we have an unavoidable array, by Theorem 4.1, but, as made clear in Figure 8, there is one other array that is unavoidable. However, $c_{3,1}$ is the only exception to the rule, as is easily checked.

1,2		
	1,3	
		1

(a) $c_{3,1}$

1,2		
	1,2	
		3

(b) $c_{3,3}, d_{3,3}$

Figure 8: 3×3 unavoidable multiple entry arrays with entries only on the diagonal.

4.3. **Multiple entry 3×3 arrays.** For $n = 3$, the list of all unavoidable multiple entry arrays, produced by case analysis by hand, is presented in Figure 9. This case analysis isn't too demanding, if one starts with fixing symbols 1 and 2 in cell $(1, 1)$, and observes that this implies that no minimal unavoidable array can ever hold another forbidden 3 in some other cell in the first row or column.

1,2,3		

(a) $a_{3,1}$

1,2	1,2	

(b) $a_{3,2}$

1,2		
	3	3

(c) $b_{3,1}, d_{3,2}$

1,2		
	1,3	
		1

(d) $c_{3,1}$

1,2	1	
	3	
		2

(e) $c_{3,2}$

1,2		
	1,2	
		3

(f) $c_{3,3}, d_{3,3}$

1,2		
	1	2
	2	1

(g) $s_{3,1}$

1,2	1	
2	3	

(h) $x_{3,1}$

1,2	1	
	1	2

(i) $x_{3,2}$

1,2	1	
	1,3	

(j) $y_{3,1}$

1,2	1	
2		
	1	

(k) $y_{3,2}$

1,2	1	
	3	
1		

(l) $z_{3,1}$

1,2	1	
1		
		2

(m) $z_{3,2}$

1,2		1
	3	
2		

(n) $w_{3,1}$

1,2		1
	3	
		1

(o) $w_{3,2}$

Figure 9: All minimal unavoidable 3×3 multiple entry arrays.

The arrays in Figure 9 are labelled such that lower case a, b, c, d, s are conjugates of arrays labelled with upper case A, B, C, D, S. The rest of the arrays are paired as conjugates, so that for instance $x_{3,1}$ is conjugate to $x_{3,2}$.

4.4. **Multiple entry 4×4 arrays on two symbols.** Figures 10 through 13 show all nonisotopic 4×4 multiple entry arrays on two symbols, i.e. where there is at least one cell in which both symbols are forbidden. The list was generated by computer in the following manner:

First, the set of all avoidable 1-symbol arrays (with the symbol 1) were generated, and reduced with respect to isotopism using Nauty. Next, for each of the generated 1-symbol arrays, forbidden 2s were added, in each case fewer than or equal to the number of already present 1s. Another run of Nauty reduced this list. This final reduced list was then tested for avoidability and minimality using UMSAT.

4.4.1. *Type I unavoidable multiple entry* 4×4 *arrays on two symbols.* The alert reader will notice that the mechanism that makes all the arrays in Figure 10 unavoidable is that there exists a single cell in which the forbidden 1s force us to place a 1, and the forbidden 2s then forces the use of a 2 in the same cell.

1,2	1,2	1,2	

1	1	1	
1,2	1,2		2

1	1	1,2	
1	1		
		2	
		2	

1	1	1	
1,2	1,2		
2	2	2	

1	1	1	
1	1,2		2
	2	2	2

1	1,2	1,2	
1	1,2		
		2	
		2	

Figure 10: All type I unavoidable 4×4 multiple entry arrays on two symbols.

4.4.2. *Type II unavoidable multiple entry* 4×4 *arrays on two symbols.* In Figure 11, the list continues, but the mechanism is now a set of two cells in each of which the forbidden 1s force us to place a 1, and the forbidden 2s force us to place at least one 2 in these cells.

Arrays of types I and II can be described in full for any array size by applying Theorem 2.4.

4.4.3. *Type III unavoidable multiple entry* 4×4 *arrays on two symbols.* The unavoidable arrays in Figure 12 are a bit more complicated. They fall under neither of the two above descriptions, but instead, there is some essential interplay between the two symbols in forcing the use of certain symbols in certain cells, eventually leading to a conflict. If we take the first of the arrays in Figure 12 as an example, we see that a 1 is forced in cell $(1, 4)$, then 2s are forced in cell $(3, 1)$ and $(4, 4)$, then a 1 is forced in $(3, 1)$ and consecutively in $(2, 3)$. After this, a 2 is forced in $(4, 3)$, which is in conflict with the 2 already forced in $(4, 4)$.

1	1,2	1,2	
1	1,2		2

1	1	1,2	
1	1		2
		2	2

1	1	1	
1,2	1,2		
1,2		2	

1	1	1	
1,2	1,2		
1			
	2	2	

1,2	1	1,2	
1,2	1		
1			
		2	

1	1	1	
1,2	1		
1,2		2	
		2	

1	1	1	
1,2	1		2
1			
		2	2

1,2	1,2	1	
1			
1			
	2		2

Figure 11: All type II unavoidable 4×4 multiple entry arrays on two symbols.

1	1	1,2	
1			2
	2	1,2	2

1,2	1,2	1	
1,2			2
	1		
2			

1,2	1,2	1	
1,2			2
	1		2

Figure 12: All type III unavoidable 4×4 multiple entry arrays on two symbols.

If we apply the same typology to the list of unavoidable multiple entry arrays in Figure 9, we see that arrays $a_{3,2}$ and $y_{3,2}$ are of type I, arrays $x_{3,2}$ and $z_{3,2}$ of type II and the array $s_{3,1}$ does not belong to any of the three types.

4.4.4. *The two remaining unavoidable multiple entry* 4×4 *arrays on two symbols.* We will return to $S_{4,\{10\}}$ in the section on fractional relaxations. For now, we only note that no single symbol is forced in place in any cell, and that there is no obvious mechanism that makes $S_{4,\{10\}}$ unavoidable.

1,2	1,2		
1,2		1,2	
	1,2	1,2	

(a) $S_{4,G}$

1,2	1,2		
1,2		1	2
	1,2	2	1

(b) $S_{4,\{10\}}$

Figure 13: The two remaining unavoidable 4×4 multiple entry arrays on two symbols.

Regarding $S_{4,G}$, however, there is an interesting thing to note, namely that it has only the entries $\{1,2\}$ in each non-empty cell. Since we only need to find a diagonal of 1s and a diagonal of 2s in order to be sure that an array on symbols 1 and 2 is avoidable, in this case we can rephrase the problem as finding a 2-factor (a 2-regular subgraph) in a certain bipartite graph. The fact that only $\{1,2\}$ occurs as entries in the array means that the edges are either free to use for either diagonal, or that they are not available at all.

1,2	1,2		
1,2		1,2	
	1,2	1,2	

$\longrightarrow$

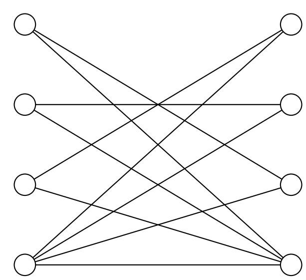

Figure 14: The type III unavoidable 4×4 multiple entry array on two symbols.

When this reformulation is possible, which is the case exactly when all non-empty cells have exactly the same entries, we can use results from graph theory to characterise the unavoidable arrays. For the 2-factor case (i.e. when all entries are $\{1,2\}$) we have the following theorem from [12], where $N(S)$ denotes the neighbour set of the set $S \subset V$.

Theorem 4.2. *Let $B = (V, E)$ be a bipartite graph. Then B has a 2-factor if and only if $|N(S)| \geq 2|S|$ for each independent set S.*

For arrays with all entries $\{1, \dots k\}$, which in the graph theoretical formulation amounts to finding k-factors in balanced bipartite graphs, we refer the reader to [11].

4.5. **Constructing unavoidable multiple arrays of arbitrary order.** Given an $n \times n$ unavoidable array A on t symbols, there is a simple construction of an unavoidable $(n + r) \times (n + r)$ multiple entry array for any $t \geq r$. The construction is illustrated in Figure 15, where B signifies an $r \times r$ array that is entirely empty, and C signifies an $r \times n$ array, where symbols $1, \ldots, r$ are forbidden in each cell. As an illustrating example, the array $S_{4,\{10\}}$ padded with two rows is presented in the same figure.

B	C
	A

		1,2	1,2	1,2	1,2
		1,2	1,2	1,2	1,2
		1,2	1,2		
		1,2		1	2
			1,2	2	1

Figure 15: The general form of a padding construction, and $S_{4,\{10\}}$ padded with two rows.

The added symbols in the subarray C can be distributed more evenly by placing half of them, or rather $\lceil \frac{r}{2} \rceil$ of them, in C, and the rest of them in the subarray below B. For instance, an unavoidable 2-symbol single entry array can be padded in such a way that it is still a single entry array. This may seem to contradict Theorem 2.4, but the construction actually again yields an unavoidable array with the structure described in that theorem.

5. Concluding remarks

Why do the sporadic unavoidable arrays of order 3 and 4 exist? They seem to break the nice pattern the other unavoidable single entry arrays make out. Is the key ingredient perhaps that they use (almost) all available symbols, each one a large number of times, or do they exist because 3 and 4 are such small numbers? In the opinion of the present authors, the second explanation is the correct one. For instance, we believe the list of arrays in Figure 16 to be a complete list of minimal unavoidable 5×5 arrays on at least three symbols, up to isotopism.

In general, we would like to propose the following conjecture.

Conjecture. *For $n \geq 5$, the following is a complete (up to isotopism) list of minimal unavoidable $n \times n$ single entry arrays:*

$$A_{n,1}, \ldots, A_{n,\lceil n/2 \rceil}, B_{n,1}, B_{n,3}, \ldots, B_{n,\lceil n/2 \rceil}, C_{n,1}, C_{n,2}, C_{n,3}, D_{n,4}, \ldots, D_{n,n}$$

1	2			3
1	2			3
1	2			3
1		1	1	
	2	2	2	

(a) $C_{5,1}$

1	1	1	1	
1				3
1				3
1				3
	2	2	2	

(b) $C_{5,2}$

1	2			
1	2			
1	2			
1	2			
		3	3	3

(c) $C_{5,3}, D_{5,3}$

1	2	3		
1	2	3		
1	2	3		
1	2	3		
			4	4

(d) $D_{5,4}$

1	2	3	4	
1	2	3	4	
1	2	3	4	
1	2	3	4	
				5

(e) $D_{5,5}$

Figure 16: Minimal unavoidable 5×5 arrays on at least 3 symbols, that belong to known families.

In particular, we conjecture that any minimal unavoidable array on at least $k \geq 5$ symbols is a $D_{n,k}$. A more tractable problem might be the following: if the only minimal unavoidable arrays on k symbols are the $D_{n,k}$, show that the same holds for $k+1 \leq n$.

Conjecture 1 may seem rash, and we concede that it is based mainly on the fact that no other families of unavoidable single entry arrays are known. As mentioned in the introduction, however, there are phenomena that provably only occur for very small arrays, for instance unavoidable partial Latin squares, which exist for orders 2 and 3 only (see [2, 3, 4]). It is not unreasonable that a similar result might hold for unavoidable single entry arrays in general. Another instance where small arrays cause problems, but larger arrays are more well-behaved is row-latin squares with empty last row. Häggkvist [8] found a single unavoidable row-latin square with empty last row, labelled $S_{3,1}$ in the present article, and proved that no such arrays exist for $n = 2^k$. In other words, there are other phenomena in this area of research where there exist some anomalous small cases, but for larger size arrays, everything works as expected. Conjecture 1 is therefore not as poorly grounded as one might believe at first glance.

By Proposition 2.1 and Theorem 2.4, all unavoidable single entry arrays on one or two symbols are characterised. Recently, it was confirmed that the arrays of type C are the only unavoidable 3-symbol arrays of order 5 or greater [1]. Hopefully, this issue will be revisited soon.

Where multiple entry arrays are concerned, the general problem of characterising all unavoidable arrays is certainly an intractable problem, but we are convinced that there are some interesting polynomial special instances. We note also that Proposition 2.1 and Theorem 2.4 yield polynomial time algorithms for recognising unavoidable arrays on one or two symbols.

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