CYCLES, WHEELS, AND GEARS IN FINITE PLANES

JAMIE PEABODY, OSCAR VEGA, AND JORDAN WHITE

Abstract. The existence of a primitive element of \( GF(q) \) with certain properties is used to prove that all cycles that could theoretically be embedded in \( AG(2,q) \) and \( PG(2,q) \) can, in fact, be embedded there (i.e. these planes are ‘pancyclic’). We also study embeddings of wheel and gear graphs in arbitrary projective planes.

1. Introduction

In this article, a graph will be understood to be simple, finite, and undirected. Since we will mostly focus on cycles and cycle-related graphs we define, for \( k \geq 3 \), a \( k \)-cycle as the graph \( C_k \) with \( V = \{x_1, \ldots, x_k\} \) and \( E = \{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1\} \). We refer the reader to [11] for any graph theoretical notion we use and fail to define.

Next, we define the concepts in finite geometry that we will need later on; any those missing concepts may be found in [2].

Definition 1.1. Let \( \pi = (P, L, I) \) where \( P \) is a set of points, \( L \) is a set of lines, and \( I \) is an incidence relation. Then \( \pi \) is an affine plane if it satisfies the following conditions:

1. Given any two distinct points, there is exactly one line incident with both of them.
2. For every line \( l \) and every point \( P \) not incident with \( l \) there is a unique line \( m \) that is incident with \( P \) and that does not intersect \( l \).
3. There are three points that do not lie on the same line.

We may obtain a projective plane \( \Pi \) from any given affine plane \( \pi \) by the addition of a line at infinity, denoted \( \ell_\infty \). Furthermore, lines which were parallel with one another in \( \pi \), meet at a point at infinity in \( \Pi \). Finally, these points at infinity are all incident with the line at infinity. Conversely, deleting any line in a projective plane (and all points incident with that line) yields an affine plane.

Received by the editors November 26, 2011, and in revised form July 4, 2013.
2010 Mathematics Subject Classification. Primary 05; Secondary 51.
Key words and phrases. Graph embeddings, finite projective plane, primitive element.

This work was supported by NSF Grant #DMS-1156273 (Fresno State’s 2012 Summer REU), and the McNair Program at Fresno State.
In this work we consider planes that contain only a finite number of points and lines. In this case, it is known that for every affine plane $\pi = (P, L, I)$ there is a positive integer $q$, called the order of the plane, such that $|P| = q^2$, $|L| = q^2 + q$, each line contains exactly $q$ points, and every point is incident with exactly $q + 1$ lines. A similar result is also valid for projective planes. In this case, the addition of $\ell_\infty$ yields the following: $|P| = |L| = q^2 + q + 1$, every line contains $q + 1$ points, and every point is incident with $q + 1$ lines. All known examples of finite planes have order equal to the power of a prime number.

It is known that for every $q$, a power of a prime, there is only one affine/projective plane that may be coordinatized by $GF(q)$. For every fixed $q$, we denote this plane by $AG(2, q)$ (if affine) or $PG(2, q)$ (if projective).

Our objective is to study how cycles, and some cycle-related graphs can be embedded in finite planes (both affine and projective). For this, we must define what we understand an embedding of a graph into a finite plane.

**Definition 1.2.** Let $G = (V, E)$ be a graph. An embedding of $G$ into a plane (affine or projective) $\pi = (P, L, I)$, is an injective function $\psi : V \rightarrow P$ that induces naturally an injective function $\overline{\psi} : E \rightarrow L$ by preserving incidence. We call $\psi$ an embedding of $G$ in $\pi$. If such a function exists, we say that $G$ embeds in $\pi$ and write $G \hookrightarrow \pi$.

Note that since $\overline{\psi}$ is injective, we will identify edges in $G$ with whole lines. That is, if a line has been used as an edge for a graph, this line cannot be used again in the same embedding.

**Definition 1.3.** We say that $AG(2, q)$ is pancyclic if and only if $C_k \hookrightarrow AG(2, q)$, for all $3 \leq k \leq q^2$. Similarly, we say that $PG(2, q)$ is pancyclic if and only if $C_k \hookrightarrow PG(2, q)$, for all $3 \leq k \leq q^2 + q + 1$.

The idea of embedding a graph into other structures has been present for a long time. For instance, the history of embeddings of graphs into linear spaces goes back to Hall [4], includes Erdős [3], and the more recent work by Moorhouse and Williford [7]. On the other hand, not much is known about embeddings of graphs in finite planes: most of what is known is on embeddings of cycles. This is likely because studying $k$-cycles embedded in a projective plane $\Pi$ is equivalent to studying embeddings of $(2k)$-cycles in the Levi graph of $\Pi$. For instance, one can use the Singer cycle in $PG(2, q)$ to construct a $(q^2 + q + 1)$-cycle in $PG(2, q)$ (e.g., see [5]). Also, the constructions by Schmeichel [8] proved that $PG(2, p)$ is pancyclic for $p$ prime. Moreover, Schmeichel’s longest cycle is different from the one constructed using the Singer cycle (these cycles are constructed in [5]). Recently, in [5], one may find expressions for the number of $k$-cycles in a projective plane of order $q$, for $3 \leq k \leq 6$. This work has been extended by Voropaev [10] to $7 \leq k \leq 10$.

Our work may also be related to [6], as in that article embeddings of cycles in projective planes are also studied. However, the approach in [6] is purely
geometrical, and our approach is an effort to bring an algebraic perspective to the problem. In fact, we will coordinatize $AG(2, q)$ using a field to give an algebraic characterization of the pancyclicity of $AG(2, q)$ and $PG(2, q)$. We refer the reader to [5] and [6] for a thorough historical narrative on this problem.

2. Cycles in $AG(2, q)$ and $PG(2, q)$

In this section we investigate pancyclicity in $AG(2, q)$ and $PG(2, q)$ by modifying the approach of constructing of cycles in [6].

Let $\mathbb{F} = GF(q)$ and let $\langle \alpha \rangle = \mathbb{F}^*$, that is, $\alpha$ is a primitive element of $\mathbb{F}$. We will consider the following coordinatization of $AG(2, q)$ using $\mathbb{F} \times \mathbb{F}$. The points on its axes will be labeled using 0 or powers of $\alpha$. Next we label the lines through $O = (0, 0)$ as follows:

$$l_i : \begin{cases} x = 0, & \text{if } i = 0 \\ y = x\alpha^i, & \text{if } i = 1, 2, \ldots, q - 1 \\ y = 0, & \text{if } i = q. \end{cases}$$

Also, for any point $Q$ in the plane we denote the line parallel to $l_i$ that passes through $Q$ by $l_i + Q$.

Pick any point $P_0 \in l_0$, different from $O$. We define $P_i = (l_{i+1} + P_{i-1}) \cap l_i$ for $i = 1, \ldots, q - 1$ and $P_q = (l_0 + P_{q-1}) \cap l_q$. Next, we connect $P_{i-1}$ with $P_i$ using $l_{i+1} + P_{i-1}$ for $i = 1, \ldots, q - 1$, and connect $P_{q-1}$ with $P_q$ using $l_0 + P_{q-1}$. In this way, we obtain a path of length $q + 1$. We denote this path by $P_{P_0}$.

In [6], it is shown that the $q - 1$ paths constructed this way share no points or lines with $P_0$ being any point on $l_0$ different from $O$. Hence, these paths partition the points of $AG(2, q) \setminus \{O\}$. Moreover, a path starting at $(0, \alpha)$ may be obtained from the path starting at $(0, 0)$ by using a translation $T_v$ with $v = (0, \beta - \alpha)$.

Note that no line parallel to $l_1$ has been used in the construction of these paths. Hence, using the line $l_1 + P_q$ we connect $P_q$ with a (uniquely determined) point $Q_0$ on $l_0$. If $Q_0 = P_0$, then we get a cycle of length $q + 1$. On the other hand, if $Q_0 \neq P_0$ then we may concatenate $P_{Q_0}$ to the path starting at $P_0$ and ending at $Q_0$ to form a longer path. It seems that when $P_0 \neq Q_0$ we are able to create long cycles. But, how long? To answer this question we need to study the case when $P_0 \neq Q_0$. We will do this algebraically by identifying $AG(2, q)$ with $\mathbb{F} \times \mathbb{F}$.

**Lemma 2.1.** Let $P_0 = (0, \beta)$ be a point in $AG(2, q)$ then

$$P_{i+1} = (y = \alpha^{i+2}x + \beta(1 + \alpha)^i) \cap (y = \alpha^{i+1}x) = \left( \frac{\beta(1 + \alpha)^i}{\alpha^{i+1}(1 - \alpha)}, \frac{\beta(1 + \alpha)^i}{1 - \alpha} \right)$$

for all $0 \leq i \leq (q - 2)$. Also,

$$P_q = \left( \frac{\beta}{(1 + \alpha)^2(1 - \alpha)}, 0 \right), \quad Q_0 = \left( 0, \frac{-\alpha\beta}{(1 - \alpha)(1 + \alpha)^2} \right).$$
Proof. Let \( P_0 = (0, \beta) \), then
\[
P_1 = (y = \alpha^2 + \beta) \cap (y = \alpha x) = \left( \frac{\beta}{\alpha(1 - \alpha)}, \frac{\beta}{(1 - \alpha)} \right).
\]
In general,
\[
P_{i+1} = (y = \alpha^{i+2}x + b) \cap (y = \alpha^{i+1}x).
\]
We find \( b \) by substituting the coordinates of \( P_i \) into \( y = \alpha^{i+2}x + b \) and get
\[
b = \beta(1 + \alpha)^{i-1} \left( \frac{\alpha^i - \alpha^{i+2}}{\alpha^i(1 - \alpha)} \right) = \beta(1 + \alpha)^i.
\]
So,
\[
P_{i+1} = (y = \alpha^{i+2}x + \beta(1 + \alpha^i)) \cap (y = \alpha^{i+1}x).
\]
We then isolate \( x \) to get
\[
x = \frac{\beta(1 + \alpha^i)}{\alpha^{i+1}(1 - \alpha)},
\]
and it follows that for \( 1 \leq i \leq (q - 2) \),
\[
P_{i+1} = \left( \frac{\beta(1 + \alpha)^{i+1}}{\alpha^{i+1}(1 - \alpha)}, \frac{\beta(1 + \alpha)^i}{(1 - \alpha)} \right).
\]
Finally, \( P_q \) and \( Q_0 \) are obtained using similar procedures. \( \square \)

The previous lemma proves that each of the paths of the form \( P_{P_0} \) starts at \( (0, \beta) \) and returns to \( l_0 \) at
\[
\left( 0, \frac{-\alpha \beta}{(1 - \alpha)(1 + \alpha)^2} \right).
\]
Note that this behavior is being dictated by the action of the group \( \mathbb{F}^* \) on itself defined by
\[
\alpha \cdot \beta = \frac{-\alpha}{(1 - \alpha)(1 + \alpha)^2} \beta.
\]
The following result is almost immediate.

**Theorem 2.2.** Assume that there is a primitive element \( \alpha \in \mathbb{F} \) such that
\[
\gamma = \frac{-\alpha}{(1 - \alpha)(1 + \alpha)^2}
\]
is also primitive. Then, \( C_{q^2-1} \hookrightarrow AG(2, q) \).

Proof. Having \( \gamma \) be primitive means that the sequential action of \( \alpha \) on \( \mathbb{F}^* \) yields the cycle
\[
\beta \xrightarrow{\alpha} \gamma \beta \xrightarrow{\alpha} \gamma^2 \beta \xrightarrow{\alpha} \ldots \xrightarrow{\alpha} \gamma^{q-2} \beta \xrightarrow{\alpha} \gamma^{q-1} \beta = \beta
\]
which runs through all the elements in \( \mathbb{F}^* \). Hence, the paths of the form \( P_{P_0} \) create a cycle of length \( q^2 - 1 \) when connected using lines parallel to \( l_1 \). \( \square \)
We have not been able to prove that the hypothesis in Theorem 2.2 must hold. However, Mathematica was used to verify that the hypothesis holds for all finite fields of order at most $10^6$ and Python was used to verify that it holds for fields of prime order $10^6 \leq p \leq 10^7$. Different labelings of the lines through $O$ in $AG(2,q)$ might yield different possibilities for $\gamma$ in Theorem 2.2. For instance,

$$l_i : \begin{cases} 
  x = 0, & \text{if } i = 0 \\
  y = \alpha^i x, & \text{if } 1 \leq i \leq q - 2 \\
  y = 0, & \text{if } i = q - 1 \\
  y = x, & \text{if } i = q \n \end{cases} \quad (2.1)$$

yields

$$\gamma' = \frac{\alpha - 1}{(\alpha + 1)^3}.$$ 

We tried many different labelings on the lines through $O$, no other interesting $\gamma$’s were found.

If $\gamma$ obtained in Theorem 2.2 were equal to $\gamma'$ obtained from (2.1), we would get $3\alpha = 1$, which would mean that $\text{char}(F) \neq 3$. Moreover, if we also assume $\gamma' = 1$, then $(\alpha + 1)^3 = \alpha - 1$, which implies $\alpha^4 = -1$, and thus $\langle \alpha \rangle$ has order 8, forcing $F = GF(9)$. However this is impossible because $\text{char}(F) \neq 3$. Hence, $\gamma \neq 1$ or $\gamma' \neq 1$. It follows that one may choose a coordinatization of $AG(2,q)$ such that $P_0 \neq Q_0$, and thus our construction may always be assumed to create cycles of length at least $2(q + 1)$.

**Lemma 2.3.** Let $q = 2^a$, where $a \in \mathbb{N}$, $a > 1$, and $\gamma'$ is the element from (2.1). Then there is a primitive element $\alpha$ of $F = GF(q)$ such that $\gamma'$ is also a primitive element.

**Proof.** When $q$ is even we obtain,

$$\gamma' = \frac{1}{(\alpha + 1)^2}.$$ 

A conjecture of Golomb that has been verified asserts that, if $q$ is even and larger than 2, then there are consecutive primitive elements $\alpha$ and $\alpha + 1$ of $F$ (see e.g., the survey [1]). Next, since $q - 1$ is odd then $(\alpha + 1)^2$ is a primitive element because $\alpha + 1$ is a primitive element and $\text{gcd}(2, q - 1) = 1$. Finally, we use the fact that the inverse of a primitive element is also a primitive element to get that $1/(\alpha + 1)^2$ is a primitive element. \qed

**Remark.** We will say that Hypothesis $J$ holds when $q$ is a power of 2, or the hypothesis in Theorem 2.2 holds.

As of now, we know that Hypothesis $J$ holds when either $q = 2^a$ for some $a \in \mathbb{N}$, $q$ is an odd prime less than $10^7$, or when $q$ is a power of an odd prime that is less than $10^6$.

The following corollary is immediate.
Corollary 2.4. If Hypothesis J holds, then $C_{q^2-1} \hookrightarrow AG(2,q)$.

The largest possible cycle that could be embedded in $AG(2,q)$ has length $q^2$. We will construct such a cycle in $AG(2,q)$ by noticing that the $q^2 - 1$ cycles already constructed do not use any of the lines through $O$.

Corollary 2.5. If Hypothesis J holds, then $C_{q^2} \hookrightarrow AG(2,q)$.

Proof. Let $P$ and $Q$ be two points in $AG(2,q)$ that are adjacent in the embedding of $C_{q^2-1}$ in $AG(2,q)$ described in Corollary 2.4. Without loss of generality, assume that $P \in l_0$ and $Q \in l_1$. We disconnect $P$ and $Q$ by eliminating the line that joins them, and then we connect each one of them with $O$ by using $l_0$ and $l_1$. This new cycle has length $q^2$.

As of now, we have proven the embedding of cycles of length $q^2 - 1$ and $q^2$. What about shorter cycles?

Theorem 2.6. If Hypothesis J holds, then $AG(2,q)$ is pancyclic.

Proof. We only need to prove $C_k \hookrightarrow AG(2,q)$ for all $3 \leq k \leq q^2 - 2$. We know (see [5]) that $K_{q+1} \hookrightarrow AG(2,q)$, and thus get $C_k \hookrightarrow AG(2,q)$ for all $3 \leq k \leq q + 1$. For $q + 2 \leq k \leq q^2 - 2$, let

$$P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_{q^2-3} \rightarrow P_{q^2-2} \rightarrow P_0$$

be the $(q^2 - 1)$-cycle in Corollary 2.4.

Let $k - 1 = (q + 1)\lambda + r$, where $r, \lambda \in \mathbb{N}$ and $0 \leq r < q + 1$. We have two cases:

1. If $r \neq 0$ then the vertices $P_1$ and $P_{k-1}$ are not on the same line through $O$. Hence the cycle

$$O \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_{k-2} \rightarrow P_{k-1} \rightarrow O$$

is a $k$-cycle embedded in $AG(2,q)$.

2. If $k - 1 = (q + 1)\lambda$ then $O$ and the vertices $P_1$ and $P_{k-1}$ are collinear.

As $q > 1$ neither $P_{k-3}$ nor $P_{k-2}$ are on the line joining $P_1$ and $O$. Note that $O$, $P_{k-2}$, and $P_{(k-3)+(q+1)+1}$ are collinear.

Since $k \leq q^2 - 2$, then $\lambda \leq q - 2$, and thus $(k - 3) + (q + 1) + 1 \leq q^2 - 1$. Hence, the cycle

$$O \rightarrow P_1 \rightarrow \cdots \rightarrow P_{k-3} \rightarrow P_{(k-3)+(q+1)} \rightarrow P_{(k-3)+(q+1)+1} \rightarrow O$$

is an embedding of $C_k$ in $AG(2,q)$.

We now prove a result equivalent to Theorem 2.6 for projective planes. Since $PG(2,q)$ is constructed from $AG(2,q)$, Theorem 2.6 also holds in $PG(2,q)$. It is also known that $C_{q^2+q+1}$ embeds in $PG(2,q)$, this cycle is constructed from the Singer cycle of the plane (see [5] and [9]). For the pancyclicity of $PG(2,q)$, it remains to embed $k$-cycles with length $q^2 \leq k \leq q^2 + q$. Our plan is to modify the embedding of $C_{q^2-1}$ in $AG(2,q)$ described in Corollary 2.4.
First take the \((q^2 - 1)\)-cycle embedded in \(PG(2,q)\) described in Corollary 2.4 and shorten it to get the following path on \(q^2 - q - 1\) vertices:

\[
P : P_1 \to P_2 \to \cdots \to P_{q^2-q-2} \to P_{q^2-q-1}.
\]

Note that the \(q + 1\) affine points \(P_{q^2-q}, P_{q^2-q+1}, \ldots, P_{q^2-2}\), and \(P_0\) have not been used in this path. Since \(P_{q^2-q} \in l_2, P_{q^2-q+1} \in l_3, \ldots, P_{q^2-2} \in l_q, \) and \(P_0 \in l_0\), we will re-label these points (to simplify the notation later) as follows:

\[
P_{q^2-q} = Q_2, P_{q^2-q+1} = Q_3, \ldots, P_{q^2-2} = Q_q, P_0 = Q_0.
\]

Hence, \(Q_i \in l_i\) for all \(i = 2, 3, \ldots, q, 0\). The points of \(PG(2,q)\) not used in this path are the:

- (a) \(q + 1\) points on \(\ell_\infty\): \{\((0), (1), \ldots, (q-1), (q)\}\}, where \((i)\) is the point on \(\ell_\infty\) incident with \(l_i\), for all \(i = 0, 1, \ldots, q\);
- (b) \(q + 1\) affine points \(Q_2, Q_3, \ldots, Q_q, Q_0,\) and \(O\).

In terms of lines, we have not used the:

- (i) line \(\ell_\infty\);
- (ii) \(q + 1\) lines through \(O\);
- (iii) \(q - 1\) lines \(m_i\) joining \(Q_i\) and \(Q_{i+1 \mod q+1}\), for all \(i = 2, 3, \ldots, q;\)
- (iv) line \(m_0\) connecting \(Q_0\) with \(P_1\), and the line \(m\) joining \(P_{q^2-q-1}\) and \(Q_2\).

This yields \(q + 1\) lines not incident with \(O\) and we are now ready to prove pancyclicity in \(PG(2,q)\).

**Theorem 2.7.** If Hypothesis J holds, then \(PG(2,q)\) is pancyclic.

**Proof.** We will use the path and information just described. This is based on the cycle described in Corollary 2.4. Recall that we only need to construct \(k\)-cycles, for \(q^2 \leq k \leq q^2 + q\).

Note that \(m_i\) is parallel to \(l_{i+2 \mod q+1}\), and thus goes through \((i + 2 \mod q + 1)\) for \(i = 2, 3, \ldots, q\). Similarly, \(m_0\) goes through \((2)\), and the line joining \(m\) is incident with \((3)\). Now, the following two paths:

\[
Q_2 \xrightarrow{l_2} (2) \xrightarrow{\ell_\infty} (3) \xrightarrow{l_3} Q_3 \xrightarrow{m_2} (4) \xrightarrow{l_4} \cdots \xrightarrow{m_{q-1}} (q) \xrightarrow{l_q} Q_q \xrightarrow{m_q} Q_0
\]

and

\[
Q_0 \xrightarrow{m_0} P_1 \to \cdots \to P_{q^2-q-1} \xrightarrow{m} Q_2
\]

may be joined to create a \((q^2 + q - 1)\)-cycle in \(PG(2,q)\). Call this cycle \(C\).

Since \(l_0, l_1,\) and \(O\) have not been used in \(C\), a slight modification of \(C\) allows us to get a \((q^2 + q)\)-cycle. That cycle is:

\[
Q_2 \xrightarrow{l_2} (2) \xrightarrow{\ell_\infty} (3) \xrightarrow{l_3} \cdots \xrightarrow{m_q} Q_0 \xrightarrow{l_0} O \xrightarrow{l_1} P_1 \to \cdots \to P_{q^2-q-1} \xrightarrow{m} Q_2.
\]

It remains to construct \(k\)-cycles for \(q^2 \leq k \leq q^2 + q - 2\).
First notice that if instead of the subpath
\[ P_{q^2-1} \rightarrow Q_2 \rightarrow (2) \rightarrow (3) \rightarrow Q_3, \]
in \(C\), we had
\[ P_{q^2-1} \rightarrow Q_2 \rightarrow O \rightarrow Q_3, \]
then we get a \((q^2 + q - 2)\)-cycle in \(PG(2, q)\). Similarly, notice that if instead of the subpath
\[ P_{q^2-1} \rightarrow Q_2 \rightarrow (2) \rightarrow (3) \rightarrow (\ell_\infty) \rightarrow (3) \rightarrow (3) \]
in \(C\), we had
\[ P_{q^2-1} \rightarrow \ell_3 \rightarrow Q_3, \]
then we get a \((q^2 + q - 3)\)-cycle in \(PG(2, q)\) that does not use \(\ell_\infty\). Call this cycle \(C'\).

Next, for \(i = 4, \ldots, q\), delete the path
\[ (3) \rightarrow Q_3 \rightarrow \cdots \rightarrow (i) \rightarrow Q_i \]
from \(C'\) (keeping (3) and \(Q_i\) in \(C'\)) and connect both (3) and \(Q_i\) with \(O\), using \(l_3\) and \(l_i\), to get the cycle
\[ (3) \rightarrow O \rightarrow Q_i \rightarrow \cdots \rightarrow P_{q^2-1} \rightarrow (3) \]
in \(C'\) which has length \((q^2 + q - 3) - (2i - 6)\). Since \(i = 4, \ldots, q\), this yields cycles of lengths \(q^2 + q - 5, q^2 + q - 7, \ldots, q^2 - q + 3\).

Now, for each of these cycles (for each \(i = 4, \ldots, q\)), replace
\[ (3) \rightarrow O \rightarrow Q_i \]
by
\[ (3) \rightarrow (2) \rightarrow O \rightarrow Q_i \]
to get cycles with length \(q^2 + q - 4, \ldots, q^2 - q + 4\). Therefore, the only cycle left to be constructed would be a \(q^2\)-cycle, in the case that \(q = 3\). However, this case is easy to handle without using the arguments in this proof. \(\square\)

3. Wheels and Gears

The graphs studied in this section are all related to cycles in some way. Embeddings of these graphs will often rely on first finding an embedding of a specific cycle and then embedding the additional vertices and edges that make up the graph. Most of the cycles we will be interested in will be short but also have some additional desirable properties. We will focus on results for projective planes although many of the constructions are generalizable to affine planes.

Throughout this section, we will use the same notation as in the previous section, and use \(\Pi_q\) to denote a generic projective plane of order \(q\) (i.e. one that is not necessarily isomorphic to \(PG(2, q)\)).
3.1. Wheel graphs. We define the wheel graph $W_n$ to be the graph on $n + 1$ vertices formed by a cycle of length $n$ and one additional vertex, called the centre, that is adjacent to every vertex in the cycle. Hence, the centre of the wheel has degree $n$. Since no vertex of a graph embedded in $\Pi_q$ can contain a vertex of degree greater than $q + 1$, we see immediately that $n \leq q + 1$ if $W_n \hookrightarrow \Pi_q$. We now show by construction that $W_{q + 1}$ can indeed be embedded in $\Pi_q$.

**Theorem 3.1.** $W_n \hookrightarrow \Pi_q$ if and only if $3 \leq n \leq q + 1$.

**Proof.** Since having $W_n \hookrightarrow \Pi_q$ implies $n \leq q + 1$, we proceed to construct wheels for all $n \leq q + 1$.

We know that $K_{q + 1} \hookrightarrow \Pi_q$, when $q$ is odd, and $K_{q + 2} \hookrightarrow \Pi_q$, when $q$ is even (see [5]). This implies that the result is obtained except, possibly when $n = q + 1$ and $q$ is odd.

Assume $q$ is odd, let $O$ be any point of $\Pi_q$, and $\ell = \{P_1, \ldots, P_{q + 1}\}$ be any line in $\Pi_q$ not incident with $O$. Denote by $\ell_i$ the line joining $O$ and $P_i$. The points $P_1, P_3, P_5, \ldots, P_q$ are vertices on the $(q + 1)$-cycle of the wheel. The edges connecting these points and $O$ are the corresponding $\ell_i$’s. The other $(q + 1)/2$ vertices will be taken from the other $\ell_i$’s. Let $m$ be a line through $P_1$, different from $\ell$ and $\ell_1$. Now choose $Q_{2i}$ to be the point on $\ell_{2i} \cap m$ for $i = 1, \ldots, (q + 1)/2$. Note that none of the $Q_i$’s can be on $\ell$.

To create the path

$$P_1 \rightarrow Q_2 \rightarrow P_3 \rightarrow Q_4 \rightarrow \cdots \rightarrow P_q,$$

we need to show that the lines connecting $P_i$ with $Q_{i \mod q + 1}$ and $Q_j$ with $P_{j \mod q + 1}$ are all distinct. This is clear because if the line connecting $P_i$ and $Q_{i \mod q + 1}$ was equal to the line connecting $P_j$ and $Q_{j \mod q + 1}$, then $P_i$ and $P_j$ would be on this line. Therefore, either we have the trivial case, $i = j$, or this line must be $\ell$, but $\ell$ does not contain any of the $Q_i$’s. A similar argument shows that all the lines needed in this path are distinct.

Now, if we extend this path into a cycle by joining $P_q$ with $Q_{q + 1}$ as done above, we would encounter the problem of having the line joining $Q_{q + 1}$ with $P_1$ being $m$, which has already been used. So, we choose a point $T_{q + 1} \in \ell_{q + 1}$ such that the line $t$, through $T_{q + 1}$ and $P_1$, is different from $m$ and $\ell$. Since every line used in the path above goes through a point $P_i$, the lines $t$ and $s$ (joining $T_{q + 1}$ and $P_q$) are different from all others. We get the $(q + 1)$-cycle

$$P_1 \rightarrow Q_2 \rightarrow P_3 \rightarrow Q_4 \rightarrow \cdots \rightarrow P_q \rightarrow T_{q + 1} \rightarrow P_1.$$

Since, all the vertices in the cycle connect to $O$ by using different lines, the vertices $\{P_1, Q_2, P_3, Q_4, \ldots, P_q, Q_{q + 1}, O\}$ form an embedding of $W_{q + 1}$, as desired. $\square$

3.2. Gear graphs. We define a gear graph, $G_n$, to be a graph on $2n + 1$ vertices and $3n$ edges. The graph consists of a $2n$-cycle, and a centre vertex that is adjacent to every other vertex in the $(2n)$-cycle. Note that no gear graph can embed in $\Pi_2$, since the smallest gear graph has 9 edges and there
are only 7 lines in such a plane. For \( q = 3 \) and \( q = 4 \), the only possible embeddings are \( G_3 \leftrightarrow \Pi_3 \), \( G_3 \leftrightarrow \Pi_4 \), \( G_4 \leftrightarrow \Pi_4 \), and \( G_5 \leftrightarrow \Pi_4 \). These are all easy to verify, and no further details will be provided. From now on, we assume \( q > 4 \).

Since the centre of \( G_n \) has degree \( n \), we want to prove that \( G_n \leftrightarrow \Pi_q \), for all \( 3 \leq n \leq q + 1 \).

**Lemma 3.2.** \( G_n \leftrightarrow \Pi_q \), for all \( 3 \leq n \leq \lfloor (q + 1)/2 \rfloor \).

**Proof.** If \( n \) is even, then \( G_{n/2} \) is a subgraph of \( W_n \). The result follows from the fact that \( W_n \leftrightarrow \Pi_q \) for all \( 3 \leq n \leq q + 1 \) (Theorem 3.1). \( \square \)

To embed larger gears in \( \Pi_q \), we need to construct a very specific family of cycles and the corresponding gears. This will all described in the proof of our next result.

**Theorem 3.3.** \( G_n \leftrightarrow \Pi_q \), for all \( 3 \leq n \leq q \).

**Proof.** The discussion for \( q = 2, 3, 4 \) was settled at the beginning of this subsection. For \( q > 4 \), Lemma 3.2 proves the theorem for all \( 3 \leq n \leq \lfloor (q + 1)/2 \rfloor \). Also, it is easy to verify that \( G_q \leftrightarrow \Pi_5 \). For \( n > \lfloor (q + 1)/2 \rfloor \) we will give explicit constructions. First, recall that two distinct paths

\[
P_{P_0} : P_0 \to P_1 \to \ldots \to P_q \\
P_{Q_0} : Q_1 \to Q_2 \to \ldots \to Q_q
\]

constructed at the beginning of Section 2 are disjoint in terms of both points and lines as long as \( P_0 \neq Q_0 \) (two points different from \( \mathcal{O} \) on \( l_0 \)). From now on, fix \( P_0 \) and given \( n > 4 \), we will choose an appropriate \( Q_0 \) that will allow us to create a \((2n)\)-cycle out of \( P_{P_0} \) and \( P_{Q_0} \).

Let \( n \) be even, then shorten \( P_{P_0} \) to

\[
P_{P_0}' : P_0 \to P_1 \to \ldots \to P_{n-2}.
\]

Note that since \( n - 2 \leq q - 2 \), no lines parallel to \( l_0 \) or \( l_1 \) have been used in the construction of this path. Since \( q > 3 \), there are at least 2 lines through (0) different from \( l_0 + P_{n-2} \), \( l_0 + P_1 \), and \( \ell_\infty \), each of these lines intersect \( l_1 \) at a point different from \( P_1 \). Now, there are \( q - 1 \) lines through (1) different from \( l_1 + P_0 \) and \( \ell_\infty \), each of these lines intersect \( l_{n-1} \) at a point different from \( l_{n-1} \cap (l_1 + P_0) \). Choose \( Q_0 \in l_0 \) so that (the points on \( P_{Q_0} \)) \( Q_{n-1} \neq l_{n-1} \cap (l_1 + P_0) \), \( Q_2 \neq P_2 \), and \( Q_2 \notin l_0 + P_{n-2} \). We get the following \((2n)\)-cycle.

\[
(1) \underbrace{l_1 + P_0 \to P_0 \to \ldots \to P_{n-2} \to l_0 + P_{n-2}}_{\text{in } P_{P_0}} \to 0 \ldots \\
\underbrace{l_0 + Q_1 \to Q_1 \to \ldots \to Q_{n-1}}_{\text{in } P_{Q_0}} \to l_1 + Q_{n-1} \to (1).
\]

To create \( G_n \), join \( \mathcal{O} \) with \( P_0, P_2, \ldots, P_{n-2}, Q_1, Q_3, \ldots, Q_{n-1} \).
When \( n \) is odd we proceed similarly. Consider the path,
\[
\mathcal{P}_P' : P_0 \to P_1 \to \cdots \to P_{n-1}.
\]
Notice that no lines through \((0)\) or \((1)\) have been used and that \(P_{n-3}\) must be different from one of \((l_1 + P_0) \cap l_{n-3}\) and \((l_n + P_0) \cap l_{n-3}\). We will say that \(P_{n-3} \neq (l_k + P_0) \cap l_{n-3}\), where \(k\) is either 0 or \(n\). So, since \(P_{n-3} \neq (l_k + P_0) \cap l_{n-3}\) we choose \(Q_0\) so that \(Q_{n-3} = (l_k + P_0) \cap l_{n-3}\). Now observe neither \(\mathcal{P}_P\) or \(\mathcal{P}_Q\) can go through \(Q_0\). Because of \(l_0\), there are at least two lines through \(Q_0\) that have not been used so far. Let \(T\) be a point on \(l_n\) such that \(T\) is on one of the lines through \(Q_0\) that are available, and \(T \neq (l_0 + P_{n-1}) \cap l_n\). Note that
\[
m = \overrightarrow{Q_0T}
\]
cannot be equal to the lines \(l_k + P_0\), or \(l_0 + T\). We get the following \((2n)\)-cycle:
\[
Q_{n-3} \xrightarrow{l_k + P_0} P_0 \to \cdots \to P_{n-1} \xrightarrow{l_0 + P_{n-1}} (0) \xrightarrow{l_0 + T} T \xrightarrow{m} Q_0 \to \cdots \to Q_{n-3}
\]
where \(l_0 + T\) could be \(\ell_\infty\).

To create \(G_n\) we join \(O\) with \(P_0, P_2, \ldots, P_{n-1}, T, Q_1, \ldots, Q_{n-4}\). We used \(n > 4\) in order to get that \(1 \leq n - 4\), and thus that the last selection of vertices (connected with \(O\)) is well defined. \(\Box\)

Thus far, we know we can embed gear graphs, \(G_n\), where \(n\) is any integer between 3 and \(q\). The next step in embeddings of gear graphs is to determine the largest gear that can be embedded.

**Theorem 3.4.** Let \(q > 4\). Then \(G_{q+1} \hookrightarrow \Pi_q\). Furthermore, this is the largest gear that can be embedded in \(\Pi_q\).

**Proof.** First, notice that \(G_{q+1}\) is the largest possible gear that can be embedded in \(\Pi_q\) because the degree of the centre of \(G_{q+1}\) is \(q + 1\), which is the largest allowed in \(\Pi_q\).

To show that \(G_{q+1}\) actually embeds in \(\Pi_q\), we need to construct a cycle of length \(2(q + 1)\) without using \(O\) and any of the \(q + 1\) lines through \(O\). The cycle needs to be constructed in such a way that we are able to connect every other vertex of the cycle to the point \(O\). The construction of this cycle depends on the parity of the order of the plane.

**Case 1:** \((q\) is even). Assume \(q\) is even. Choose \(q + 1\) points \(P_i \in l_i \setminus \{O(i)\}\) for \(i = 0, 1, \ldots, q\). Choose \(P_1\) arbitrarily, next choose \(P_3\) such that \(P_3 \notin l_2 + P_1\), then choose \(P_5\) such that \(P_5 \notin l_4 + P_3\), etc. In general, choose \(P_i \notin l_{i-1} + P_{i-2}\) for all \(i = 1, 3, \ldots, q-1\) (only for \(i\) odd). Now choose \(P_0\) such that \(P_0 \notin l_q + P_{q-1}\). As done before, we choose \(P_i \notin l_{i-1} + P_{i-2}\) for \(i = 2, 3, \ldots, q\) (now only for \(i\) even) taking care with the choice of \(P_q\) that \(P_1 \notin l_0 + P_q\). If for the chosen \(P_q\), \(P_1 \in l_0 + P_q\) then choose a different \(P_q\). We can do this
because the only condition to choose $P_q$ was that $P_q \notin l_{q-1} + P_{q-2}$. It follows that we get the $(2q + 2)$-cycle:

$$(0) \xrightarrow{l_0 + P_1} P_1 \xrightarrow{l_2 + P_2} (2) \rightarrow \cdots \rightarrow P_{q-1} \xrightarrow{l_q + P_{q-1}} (q) \xrightarrow{l_q + P_0} P_0 \xrightarrow{l_1 + P_1} (1) \cdots \rightarrow (1) \xrightarrow{l_1 + P_2} P_2 \rightarrow \cdots \rightarrow (q - 1) \xrightarrow{l_{q-1} + P_q} P_q \xrightarrow{l_0 + P_1} (0).$$

We obtain $G_{q+1}$ by joining $O$ with $(0), (2), (4), \ldots (q), (1), (3), \ldots, (q - 1)$ using the $q + 1$ lines through $O$.

Note that this proof also works for $q = 4$.

**Case 2:** ($q$ is odd).

We perform a similar construction to the one for $q$ even. Choose $P_1, P_3, \ldots, P_q$ as above, then begin by arbitrarily choosing $P_2$, and continue in the same fashion to get $P_2, P_4, \ldots, P_{q-1}$. In case $P_1 \notin l_q + P_{q-1}$, choose a different $P_{q-1}$. Hence the line $l_q + P_1$ has not been used yet. Now we want to find a point $T$ such that $T \notin l_0 + P_q, T \notin l_1 + P_2, T \notin l_0, T \notin l_1$, and that $T \neq P_i$ for $i = 1, 2, \ldots, q$. This point will be connected to $(0)$ and $(1)$ to create the cycle we need.

There are $q - 2$ lines through $(0)$ different from $l_0 + P_q, l_0$, and $\ell_\infty$. These $q - 2$ lines may be used to connect $(0)$ with $q(q - 2)$ distinct points of $\Pi_q$. Similarly, there are $q - 2$ lines through $(1)$ different from $l_1 + P_2$, $l_1$, and $\ell_\infty$. It follows that there are at least

$$q(q - 2) - 2(q - 1) = q(q - 4) + 2 \geq q + 2$$

points that can be reached simultaneously by lines through $(0)$ or $(1)$, different from the six lines to be avoided. Of these points, at most $q$ could be a $P_i$. Hence, there are at least two possibilities to choose $T$ from and we get the following $(2q + 2)$-cycle:

$$(1) \xrightarrow{l_1 + P_2} P_2 \xrightarrow{l_3 + P_2} (3) \rightarrow \cdots \rightarrow P_{q-1} \xrightarrow{l_q + P_{q-1}} (q) \xrightarrow{l_q + P_1} P_1 \xrightarrow{l_2 + P_1} (2) \cdots \rightarrow (2) \xrightarrow{l_2 + P_3} P_3 \rightarrow \cdots \rightarrow (q - 1) \xrightarrow{l_{q-1} + P_q} P_q \xrightarrow{l_0 + P_q} (0) \xrightarrow{l_0 + T} T \xrightarrow{l_1 + T} (1).$$

We obtain $G_{q+1}$ by joining $O$ with $(1), (3), \ldots, (q), (2), (4), \ldots, (q - 1), (0)$ using the $q + 1$ lines through $O$.

We have also investigated using the techniques to study embeddings of other cycle-related families of graphs such as helm graphs and prism graphs, obtaining similar results. Current work focuses on developing a theory of embeddings in finite projective spaces.

**References**


5. Felix Lazebnik, Keith E. Mellinger, and Oscar Vega, *On the number of k-gons in finite projective planes*, Note Mat. 29 (2009), no. suppl. 1, 135–151. MR 2942764


DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY, FRESNO
FRESNO, CA 93740.
E-mail address: jpeabody09@mail.fresnostate.edu

DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY, FRESNO
FRESNO, CA 93740.
E-mail address: ovega@csufresno.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS
CALIFORNIA STATE UNIVERSITY, MONTEREY BAY
SEASIDE, CA 93955.
E-mail address: jorwhite@csumb.edu